Weyl asymptotics for non-self-adjoint operators and related questions

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Abstract

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Résumé

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Contents

1 Introduction

The original plan was to present a streamlined and simplified treatment of Weyl asymptotics for differential operators in any dimension, but unfortunately, I did yet have enough time for that project. So rather I will give an introductory treatment of pseudospectra, resolvent estimates and Weyl asymptotic, restricting myself mainly to the one dimensional case, where many of the basic ideas (valid also in higher dimension) can be explained more simply. I will also discuss the relation between resolvent bounds and semigroup bounds including some recent explicit estimates (due to B. Helffer-Sj) in the Gearhardt-Prüss theorem.

- The possibility of the resolvent to be very large far away from the spectrum, is a well-known difficulty in non-self-adjoint problems, and we start by explaining the corresponding notion of pseudospectrum.
- Then we go on by discussing estimates for the resolvent close to the boundary of "the pseudospectral region" and a related result in relation to the Gearhardt-Prüss theorem.
- The remaining part of the lectures deal with the main subject, namely that thanks to spectral instability (induced by the pseudospectrum), if we add small random perturbations, then with probability close to 1, the eigenvalues of a differential operator will distribute according to the Weyl law, well-known for self-adjoint – and more generally normal – differential operators.

To a large extent (with the exception of a recent result of W. Bordeaux-Montrieux and the explicit estimates in the Gearhardt-Prüss theorem), these notes are contained in the more extensive lecture notes [63] where Weyl asymptotics in higher dimensions can also be found.

2 Pseudospectrum, quasi-modes and spectral instability

Let $\mathcal{H}$ be a complex Hilbert space and let $P : \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator. Recall that the resolvent set is defined as

$$\rho(P) = \{z \in \mathbf{C};\ P - z : \mathcal{D}(P) \to \mathcal{H} \text{ has a bounded 2-sided inverse}\}.$$

It is an open set, if $z \in \rho(P)$ and $\|(z-P)^{-1}\| = 1/\epsilon$, then the open disc $D(z,\epsilon)$ is contained in $\rho(P)$. The spectrum of P is the closed set

$$\sigma(P) = \mathbf{C} \setminus \rho(P).$$

Following Trefethen–M. Embree [66] we define, for $\epsilon > 0$, the ϵ-pseudospectrum as the open set

$$\sigma_\epsilon(P) = \sigma(P) \cup \{z \in \rho(P);\ \|(z-P)^{-1}\| > 1/\epsilon\}. \tag{2.1}$$

Unlike the spectrum, the ϵ-pseudospectrum will change if we replace the given norm on $\mathcal{H}$ by an equivalent one.

$\sigma_\epsilon(P)$ can be characterized as a set of spectral instability, by the following simplified version of a theorem of Roch and Silberman:

$$\sigma_\epsilon(P) = \bigcup_{\substack{Q \in \mathcal{L}(\mathcal{H},\mathcal{H}) \\ \|Q\| < \epsilon}} \sigma(P+Q). \tag{2.2}$$

The result becomes more sublte if we use the more traditional definition with a non-strict inequality in (2.1).

Proof. Let $\widetilde{\sigma}_\epsilon(P)$ denote the right hand side in (2.2). If $z \in \mathbf{C} \setminus \sigma_\epsilon(P)$, then by a perturbation argument, we see that $z \in \mathbf{C} \setminus \widetilde{\sigma}_\epsilon(P)$.

Let $z \in \sigma_\epsilon(P)$. If $z \in \sigma(P)$ we also have $z \in \widetilde{\sigma}_\epsilon(P)$, so we may assume that $z \in \rho(P)$. Then $\exists\, u \in \mathcal{D}(P)$, $v \in \mathcal{H}$ such that $\|u\| = 1$, $\|v\| < \epsilon$, $(P-z)u = v$. Let Q be the rank one operator from $\mathcal{H}$ to $\mathcal{H}$, given by $Q\phi = -(\phi|u)v$. Then $\|Q\| = \|u\|\|v\| < \epsilon$ and $(P+Q-z)u = v + Qu = v - v = 0$, so $z \in \sigma(P+Q)$, and $z \in \widetilde{\sigma}_\epsilon(P)$. □

Using the subharmonicity of the function $z \mapsto \|(z-P)^{-1}\|$ we notice that every bounded connected component of $\sigma_\epsilon(P)$ contains an element of $\sigma(P)$.

We next discuss the construction of *quasimodes* for non-normal differential operators which shows that very often we get large ϵ-pseudospectra. The background and starting point is a result by E.B. Davies [14] for non-selfadjoint Schrödinger operators in dimension 1. M. Zworski [69] observed that this is essentially an old result of Hörmander [41, 42](1960), and that we have the following generalization, with $\{a,b\} = a'_\xi \cdot b'_x - a'_x \cdot b'_\xi = H_a(b)$ denoting the Poisson bracket of $a = a(x,\xi)$, $b(x,\xi)$.

Theorem 2.1 *Let*

$$P(x,hD_x) = \sum_{|\alpha| \le m} a_\alpha(x)(hD_x)^\alpha,\ D_x = \frac{1}{i}\frac{\partial}{\partial x} \tag{2.3}$$

have smooth coefficients in the open set $\Omega \subset \mathbf{R}^n$. *Put* $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$. *Assume* $z = p(x_0,\xi_0)$ *with* $\frac{1}{i}\{p,\overline{p}\}(x_0,\xi_0) > 0$. *Then* $\exists\ u = u_h$, *with* $\|u\| = 1$, $\|(P-z)u\| = \mathcal{O}(h^\infty)$, *when* $h \to 0$.

In the case when the coefficients are all analytic we can replace "h^∞" by "$e^{-1/Ch}$ for some $C>0$".

Notice that this implies that if the resolvent $(P-z)^{-1}$ exists then its norm is greater than any negative power of h when $h\to 0$ (and even exponentially large in the analytic case).

In the case $n\ge 2$, we noticed with A. Melin in [51] that if $z=p(\rho)$ and $\Re p$, $\Im p$ are independent at ρ, then $\frac{1}{i}\{p,\overline{p}\}$ times the natural Liouville measure is equal to a constant times the restriction to $p^{-1}(z)$ of σ^{n-1} which is a closed form. It follows that if Γ is a compact connected component of $p^{-1}(z)$ on which $d\Re p$ and $d\Im p$ are pointwise independent, then the average of $\frac{1}{i}\{p,\overline{p}\}$ over Γ with respect to the Liouville measure has to vanish. Hence if there is a point on Γ where the Poisson bracket is $\neq 0$ then there is also point where it is positive. In the case $n=1$ we have a similar phenomenon: If (for instance thanks to suitable ellipticity assumption) we know that $p^{-1}(z)$ is finite and that $\frac{1}{i}\{p,\overline{p}\}$ is $\neq 0$ everywhere on that set, then this set is finite and if it is contained in the interior of a connected bounded set Ω in phase space with smooth boundary such that the variation of $\arg(p-z)$ along that boundary is equal to zero, then we have to have an equal number of points in $p^{-1}(z)$ where $\frac{1}{i}\{p,\overline{p}\}$ is positive and where it is negative. This follows from the observation that the argument variation of $p-z$ along a small positively oriented circle around a point in $p^{-1}(z)$ is $\mp 2\pi$ when $\pm\frac{1}{i}\{p,\overline{p}\}>0$ at that point.

Example 2.2 $P=-h^2\Delta+V(x)$, $p(x,\xi)=\xi^2+V(x)$, $\frac{1}{i}\{p,\overline{p}\}=-4\xi\cdot\Im V'(x)$.

More recently K. Pravda-Starov [54] improved this result by adapting a more refined quasi-mode construction of R. Moyer (in 2 dimensions) and Hörmander [45] for adjoints of operators that do not satisfy the Nirenberg-Trèves condition (Ψ) for local solvability.

The proof in the C^∞-case in [69] is by a standard reduction of semi-classical results to classical results in ordinary microlocal analysis. In [19] we gave a direct proof and also treated the case of analytic coefficients, which is also essentially quite old. We refer to [63] for a simple outline of the proof of Theorem 2.1.

In the one dimensional case (that was considered by E.B. Davies) the proof is simpler. It is particularly simple in the following special case:

Consider the unbounded operator $P:L^2(S^1)\to L^2(S^1)$, $S^1=\mathbf{R}/2\pi\mathbf{Z}$ given by

$$P=hD_x+g(x) \tag{2.4}$$

where $g \in C^\infty(S^1)$ and we assume (for simplicity) that $\Im g$ has only two critical points, a unique maximum and a unique minimum. The associated symbol is $p(x,\xi) = \xi + g(x)$ on $T^*S^1 = S^1 \times \mathbf{R}$ and it is easy to see that the spectrum of P is given by an arithmetic progression of simple discrete eigenvalues situated on the line

$$\{z \in \mathbf{C};\ \Im z = \langle \Im g \rangle\}, \tag{2.5}$$

where $\langle \Im g \rangle = \frac{1}{2\pi}\int_{S^1} \Im g dx$ is the mean value of $\Im g$. This line belongs to the interior of the range of the symbol p:

$$\inf \Im g < \Im z < \sup \Im g. \tag{2.6}$$

and Theorem 2.1 applies to every z in that band. Indeed the equation $z = p(x,\xi)$ has precisely two solutions $\rho_\pm = (x_\pm, \xi_\pm)$ where $x_\pm$ are the two solutions of $\Im g = \Im z$, chosen so that $\pm \Im g'(x_\pm) < 0$ and $\xi_\pm = \Re z - \Re g(x_\pm)$. We see that $\pm\frac{1}{i}\{p, \overline{p}\}(\rho_\pm) > 0$, and a quasimode associated to ρ_+ will be given in Section 5.

3 Resolvent growth near the boundary of the pseudospectral region

Let $P \sim p + hp_1 + h^2 p_2 + ...$ in $S(m)$ on $\mathbf{R}^{2n}$, where $m \geq 1$ is an order function (see [20] for details). Assume that

$$\exists w \in \mathbf{C},\ C > 0,\ \text{such that } |p(\rho) - w| \geq m(\rho)/C,\ \forall \rho \in \mathbf{R}^{2n}.$$

Let $P = P^w(x, hD; h)$ denote the Weyl quantization of $P(x, h\xi; h)$. $\Sigma :=$ closure of the range of p, $\Sigma_\infty :=$ the set of accumulation points of $p(x,\xi)$ when $(x,\xi) \to \infty$.

Assume for simplicity that $\Re p \geq 0$ and consider a point $z_0 \in (\Sigma \backslash \Sigma_\infty) \cap i\mathbf{R}$. Using the semi-classical version of the sharp Gårding inequality we see that

$$\|(z - P)^{-1}\| \leq \mathcal{O}(\frac{1}{|\Re z|}),\ \Re z \leq -C_1 h,\ \Im z = \mathcal{O}(1).$$

Then we have the following slight improvement of one of the main results in [19]:

Theorem 3.1 *1) ([19]) If $p^{-1}(z_0)$ does not contain any maximal integral curve of $H_{\Im p}$, then $\exists\, C_0$ such that*

$$\|(z - P)^{-1}\| \leq \frac{\mathcal{O}(1)}{h} \exp(\frac{C_0 \Re z}{h}),$$

for $|\Im z - \Im z_0| \leq 1/C_0$, $-C_1 h \leq \Re z \leq \mathcal{O}(1) h \ln \frac{1}{h}$.
2) ([64]) If there exists $k \in 2\mathbf{N}$ *such that for every* $\rho \in p^{-1}(z_0)$, *we have* $H^j_{\Im p} \Re p(\rho) > 0$ *for some* $j \leq k$, *then*

$$\|(z-P)^{-1}\| \leq \frac{\mathcal{O}(1)}{h^{k/(k+1)}} \exp(C_0 \frac{(\Re z)^{(k+1)/k}}{h}),$$

for $|\Im z - \Im z_0| \leq 1/C_0$, $-h^{\frac{k}{k+1}} \leq \Re z \leq \mathcal{O}(1)(h \ln \frac{1}{h})^{\frac{k}{k+1}}$.

This theorem is proved by "constructing" $e^{-tP/h}$ as a Fourier integral operator with complex phase and obtain an estimate

$$\|e^{-tP/h}\| \leq C \exp(-\frac{t^{k+1}}{Ch})$$

and then integrating:

$$e^{-tP/h} = -\frac{1}{h} \int_0^\infty \exp \frac{t(z-P)}{h} dt.$$

Notice that when $m = 1$ and with some weak additional assumptions for more general m, we have by means of the semi-classical version of the sharp Gårding inequality,

$$\|(P-z)^{-1}\| \leq \frac{1}{(-Ch - \Re z)}, \text{ when } \Re z \leq -Ch.$$

$k = 2$, $n = 1$: There are more precise results by Almog, Almog-Helffer, J. Martinet, W. Bordeaux Montrieux. The most recent one is due to Bordeaux Montrieux:

Assume $n = 1$, $z_0 \in (\partial\Sigma) \setminus \Sigma_\infty$, $p^{-1}(z_0) = \{\rho_0\}$, $\{p, \{p, \overline{p}\}\}(\rho_0) \neq 0$. Then $\partial\Sigma$ is smooth near z_0. Let $W \Subset \mathbf{C}$ be a sufficiently small neighborhood of z_0. For $z \in W \cap \Sigma$, let $\alpha = \alpha(z) = \mathrm{dist}\,(z, \partial\Sigma)$. Then $p^{-1}(z) = \{\rho_+, \rho_-\}$, where

$$\pm \frac{1}{2i} \{p, \overline{p}\}(\rho_\pm) \asymp \alpha^{\frac{1}{2}},\ \mathrm{dist}\,(\rho_+, \rho_-) \asymp \alpha^{\frac{1}{2}}.$$

Let $\ell_0(z) = \int_\gamma \xi dx$, where γ is a suitable curve in $p^{-1}(z)$ joining ρ_- to ρ_+. Then

$$\Im \ell_0(z) \asymp \alpha^{\frac{3}{2}}.$$

Theorem 3.2 *([5]).* $\forall C_0, C_1 > 0$, $\exists C_2 > 0$ *such that for*

$$z \in W \cap \Sigma, \ \frac{h^{\frac{2}{3}}}{C_0} \leq \alpha \leq C_1 (h \ln h)^{\frac{2}{3}}, \ h < C_2^{-1} :$$

$$\|(P-z)^{-1}\| = \frac{\sqrt{\pi}\exp(\Im \ell_0(z)/h)}{h^{\frac{1}{2}}(\frac{1}{2i}\{p,\overline{p}\}(\rho_+))^{\frac{1}{4}}(\frac{1}{2i}\{\overline{p},p\}(\rho_-))^{\frac{1}{4}}}(1+\mathcal{O}(\frac{h}{\alpha^{\frac{3}{2}}})) + \mathcal{O}(\frac{1}{h^{\frac{1}{2}}\alpha^{\frac{1}{4}}}).$$

Application: precise description of the pseudospectral boundaries for the complex harmonic oscillator and more general h-differential operators in one dimension.

We believe that this result will be important in explaining some accumulation of eigenvalues to curves near the boundary of the range of p when we add random perturbations.

Open problem: Find a similar precise result when $k = 2$, $n \geq 2$.

4 Resolvents and semi-groups, the Gearhardt-Prüss theorem

Let $\mathcal{H}$ be a complex Hilbert space and let $[0, +\infty[\ni t \mapsto S(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be a strongly continuous semigroup with $S(0) = I$. Recall that by the Banach-Steinhaus theorem, $\sup_J \|S(t)\| =: m(J)$ is bounded for every compact interval $J \subset [0, +\infty[$. Using the semigroup property it follows easily that there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $S(t)$ has the property

$$P(M, \omega_0) : \quad \|S(t)\| \leq M e^{\omega_0 t}, \ t \geq 0. \tag{4.1}$$

In fact, we have this for $0 \leq t < 1$ and for larger values of t, write $t = [t] + r$, $[t] \in \mathbb{N}$, $0 \leq r < 1$, and $S(t) = S(1)^{[t]} S(r)$.

Let A be the generator of the semigroup (so that formally $S(t) = \exp tA$) and recall (cf. [21], Chapter II or [53]) that A is closed and densely defined. We also recall ([21], Theorem II.1.10) that

$$(z-A)^{-1} = \int_0^\infty S(t) e^{-tz} dt, \quad \|(z-A)^{-1}\| \leq \frac{M}{\Re z - \omega_0}, \tag{4.2}$$

when $P(M, \omega_0)$ holds and z belongs to the open half-plane $\Re z > \omega_0$.

Recall the Hille-Yoshida theorem ([21], Th. II.3.5) according to which the following three statements are equivalent when $\omega \in \mathbb{R}$:

- $P(1, \omega)$ holds.
- $\|(z-A)^{-1}\| \le (\Re z - \omega)^{-1}$, when $z \in \mathbb{C}$ and $\Re z > \omega$.
- $\|(\lambda - A)^{-1}\| \le (\lambda - \omega)^{-1}$, when $\lambda \in]\omega, +\infty[$.

Here we may notice that we get from the special case $\omega = 0$ to general ω by passing from $S(t)$ to $\widetilde{S}(t) = e^{-\omega t} S(t)$.

Also recall that there is a similar characterization of the property $P(M, \omega)$ when $M > 1$, in terms of the norms of all powers of the resolvent. This is the Feller-Miyadera-Phillips theorem ([21], Th. II.3.8). Since we need all powers of the resolvent, the practical usefulness of that result is less evident.

We next recall the Gearhardt-Prüss-Hwang-Greiner theorem, [21], Theorem V.I.11, [66], Theorem 19.1. See also [67], [17].

Theorem 4.1 *(a) Assume that $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \ge \omega$. Then there exists a constant $M > 0$ such that $P(M, \omega)$ holds.*

(b) If $P(M, \omega)$ holds, then for every $\alpha > \omega$, $\|(z-A)^{-1}\|$ is uniformly bounded in the half-plane $\Re z \ge \alpha$.

The part (b) follows from (4.2) with ω_0 replaced by ω.

The idea in the proof of part (a) is to use that the resolvent and the inhomogeneous equation $(\partial_t - A)u = w$ in exponentially weighted spaces are related via Fourier-Laplace transform and we can use Plancherel's formula. Variants of this simple idea have also been used in more concrete situations. See [8, 23, 37, 59].

Let

$$\omega_0 = \inf\{\omega \in \mathbb{R};\ \{z \in \mathbb{C}; \Re z > \omega\} \subset \rho(A) \text{ and } \sup_{\Re z > \omega} \|(z-A)^{-1}\| < \infty\}.$$

For $\omega > \omega_0$, define $r(\omega)$ by

$$\frac{1}{r(\omega)} = \sup_{\Re z > \omega} \|(z-A)^{-1}\|.$$

Then $r(\omega)$ is an increasing function of ω and using perturbative arguments for the resolvent, when varying the spectral parameter slightly, one can show that r is a Lipschitz function on $]\omega_0, +\infty[$ satisfying

$$0 \le \frac{dr}{d\omega} \le 1\,.$$

Moreover, if $\omega_0 > -\infty$, then $r(\omega) \to 0$ when $\omega \searrow \omega_0$.

The following result gives a quantitative bound for the most interesting part in Theorem 4.1:

Theorem 4.2 *(B. Helffer, Sj [32]) We make the assumptions of Theorem 4.1, (a) and define* $r(\omega) > 0$ *by*

$$\frac{1}{r(\omega)} = \sup_{\Re z \geq \omega} \|(z-A)^{-1}\|.$$

Let $m(t) \geq \|S(t)\|$ *be a continuous positive function. Then for all* $t, a, \widetilde{a} > 0$*, such that* $t = a + \widetilde{a}$*, we have*

$$\|S(t)\| \leq \frac{e^{\omega t}}{r(\omega) \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,\widetilde{a}])}}. \tag{4.3}$$

Here the norms are always the natural ones obtained from $\mathcal{H}$, L^2.

We also have the following variant of the main result that can be useful in problems of return to equilibrium.

Theorem 4.3 *([32]) We make the assumptions of Theorem 4.2, so that (4.3) holds. Let* $\widetilde{\omega} < \omega$ *and assume that* A *has no spectrum on the line* $\Re z = \widetilde{\omega}$ *and that the spectrum of* A *in the half-plane* $\Re z > \widetilde{\omega}$ *is compact (and included in the strip* $\widetilde{\omega} < \Re z < \omega$*). Assume that* $\|(z-A)^{-1}\|$ *is uniformly bounded on* $\{z \in \mathbb{C};\, \Re z \geq \widetilde{\omega}\} \setminus U$*, where* U *is any neighborhood of* $\sigma_+(A) := \{z \in \sigma(A);\, \Re z > \widetilde{\omega}\}$ *and define* $r(\widetilde{\omega})$ *by*

$$\frac{1}{r(\widetilde{\omega})} = \sup_{\Re z = \widetilde{\omega}} \|(z-A)^{-1}\|.$$

Then for every $t > 0$*,*

$$S(t) = S(t)\Pi_+ + R(t) = S(t)\Pi_+ + S(t)(1-\Pi_+),$$

where for all $a, \widetilde{a} > 0$ *with* $a + \widetilde{a} = t$*,*

$$\|R(t)\| \leq \frac{e^{\widetilde{\omega} t}}{r(\widetilde{\omega}) \|\frac{1}{m}\|_{e^{-\widetilde{\omega} \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\widetilde{\omega} \cdot} L^2([0,\widetilde{a}])}} \|I - \Pi_+\|. \tag{4.4}$$

Here Π_+ *denotes the spectral projection associated to* $\sigma_+(A)$*:*

$$\Pi_+ = \frac{1}{2\pi i} \int_{\partial V} (z-A)^{-1} dz,$$

where V *is any compact neighborhood of* $\sigma_+(A)$ *with* C^1 *boundary, disjoint from* $\sigma(A) \setminus \sigma_+(A)$*.*

The proof of Theorem 4.2, follows the general idea in the proof of the Gearhardt-Prüss theorem. On first notices that by means of Fourier-Laplace transform and Plancherel's theorem, the resolvent bounds imply an L^2 bound for the solution of the inhomogeneous equation

$$(\partial_t - A)u = w \text{ on } \mathbb{R}. \tag{4.5}$$

in spaces with linear exponential weight. To treat $\exp tA$ we use the bound $m(t)$ to control this operator for "small" t, then make a cutoff, and correct by solving the inhomogeneous equation above. Finally we have to pass from the L^2 norm of the solution to the L^∞ bound and here again we use the bound $m(t)$. For more details, see [32].

We end by giving an example that combines the theorems 3.1 and 4.2. Let $P : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ be a closed operator depending on $0 < h \ll 1$ such that

$$\|(P-z)^{-1}\| \le (-\Re z - C_0 h)^{-1}, \ \Re z < -C_0 h, \tag{4.6}$$

as is typically the case when P is an h-pseudodifferential operator with symbol p satisfying $\Re p \ge 0$ for which the semi-classical sharp Gårding inequality $\Re(Pu|u) \ge -C_0 h \|u\|^2$ is valid. Then by the Hille Yoshida theorem, P generates a semigroup $e^{-tP/h}$, $t \ge 0$ which satisfies,

$$\|e^{-tP/h}\| \le e^{C_0 t}. \tag{4.7}$$

Now assume that P also satisfies the estimate in part 2 of Theorem 3.1 for $k = 2$ (to fix the ideas) and more precisely that

$$\|(z-P)^{-1}\| \le \frac{C_1}{h^{2/3}} e^{C_2 \frac{(\Re z)^{\frac{3}{2}}}{h}}, \ -h^{\frac{2}{3}} \le \Re z \le \epsilon(h), \tag{4.8}$$

where we can take $\epsilon(h) = C(h \ln \frac{1}{h})^{2/3}$ with C arbitrarily large. However in the case of analytic pseudodifferential operators, we may have $\epsilon(h) = 1/\mathcal{O}(1)$ and in the following we shall only assume that $\epsilon(h) \le 1$.

We apply Theorem 4.2 to e^{sA}, $A = -P$, so that by (4.7)

$$\|e^{sA}\| \le e^{C_0 h s} \tag{4.9}$$

and by (4.8), (4.6) and the maximum principle,

$$\|(A-z)^{-1}\| \le \frac{C_1}{h^{\frac{2}{3}}} e^{C_2 \frac{\alpha^{\frac{3}{2}}}{h}}, \ \Re z \ge -\alpha, \tag{4.10}$$

when $0 \le \alpha \le \epsilon(h)$. Then for $s > 0$, we can take $a = \widetilde{a} = s/2$ in Theorem 4.2 and get

$$\|e^{sA}\| \le \frac{C_1 e^{C_2 \frac{\alpha^{\frac{3}{2}}}{h} - \alpha s}}{h^{\frac{2}{3}} \int_0^{\frac{s}{2}} e^{-2\sigma(C_0 h + \alpha)} d\sigma}. \tag{4.11}$$

The integral in the denominator in (4.11) is $\asymp \min(s, 1/(C_0h+\alpha))$, so we get

$$\|e^{sA}\| \le \mathcal{O}(1)\begin{cases} \frac{C_1}{h^{\frac{2}{3}}s}e^{C_2\frac{\alpha^{\frac{3}{2}}}{h}-\alpha s}, & \text{when } s \le \frac{\mathcal{O}(1)}{C_0h+\alpha}, \\ \frac{C_1(C_0h+\alpha)}{h^{\frac{2}{3}}}e^{C_2\frac{\alpha^{\frac{3}{2}}}{h}-\alpha s}, & \text{when } s \ge \frac{1}{\mathcal{O}(1)(C_0h+\alpha)} \end{cases}$$

Assume for simplicity that $C_2 = 2/3$. Then since $A = -P$:

$$\|e^{-\frac{t}{h}P}\| \le \mathcal{O}(1)\begin{cases} \frac{C_1h^{\frac{1}{3}}}{t}e^{\frac{1}{h}(\frac{2}{3}\alpha^{\frac{3}{2}}-\alpha t)}, & \text{when } t \le \frac{\mathcal{O}(h)}{C_0h+\alpha}, \\ \frac{C_1(C_0h+\alpha)}{h^{\frac{2}{3}}}e^{\frac{1}{h}(\frac{2}{3}\alpha^{\frac{3}{2}}-\alpha t)}, & \text{when } t \ge \frac{h}{\mathcal{O}(1)(C_0h+\alpha)}. \end{cases} \tag{4.12}$$

Recall that $0 \le \alpha \le \epsilon(h)$ and choose $\alpha = \alpha(t)$ in that interval so that the exponential factor is as small as possible. The α derivative of the exponent vanishes when $\alpha = t^2$.

I. If $t^2 \le \epsilon(h)$, we choose $\alpha = t^2$ and the exponential factor in (4.12) becomes $\exp -\frac{t^3}{3h}$.

   a) If $t^3 \le h$, we are in the first case in (4.12) and hence

   $$\|e^{-tP/h}\| \le \mathcal{O}(1)\frac{C_1h^{\frac{1}{3}}}{t}e^{-\frac{t^3}{3h}},$$

   and we can neglect this case since we do not obtain any improvement of (4.9).

   b) If $t \in [h^{1/3}, \epsilon(h)^{1/2}]$ (which is a non-empty interval, since $\epsilon(h) \ge h^{2/3}$), then $\alpha = t^2 \le \epsilon(h)$ and we are in the second case in (4.12), and we get

   $$\|e^{-tP/h}\| \le \mathcal{O}(1)\frac{C_1(C_0h+t^2)}{h^{\frac{2}{3}}}e^{-t^3/(3h)}.$$

II. If $t > \epsilon(h)^{1/2}(\ge h^{1/3})$, we choose $\alpha = \epsilon(h)$ and we are again in case 2 in (4.12), so we get

$$\begin{aligned}\|e^{-tP/h}\| &\le \mathcal{O}(1)C_1\frac{(C_0h+\epsilon(h)}{h^{\frac{2}{3}}}e^{\frac{1}{h}(\frac{2}{3}\epsilon(h)^{\frac{3}{2}}-\epsilon(h)t)} \\ &= \mathcal{O}(1)C_1\frac{(C_0h+\epsilon(h)}{h^{\frac{2}{3}}}e^{-\frac{1}{h}(\frac{1}{3}\epsilon(h)^{\frac{3}{2}}+\epsilon(h)(t-\epsilon(h)^{\frac{1}{2}}))}.\end{aligned}$$

As already mentioned, the idea of the proof of Theorem 3.1 is that an estimate

$$\| \exp(-tP/h)\| \leq \exp(-t^3/(3h)) \tag{4.13}$$

together with Laplace transform leads to the estimate

$$\|(z-p)^{-1}\| \leq \frac{\mathcal{O}(1)}{h^{2/3}} \exp(-\frac{2}{3h}(\Re z)^{\frac{3}{2}}),$$

for $\Re z \geq 0$ and actually the $h^{2/3}$ can be replaced by $h^{1/2}$ when $\Re z \geq 1/\mathcal{O}(1)$. Theorem 4.2 comes rather close to recovering the semi-group bound (4.13) from the resolvent estimate, and the full power of the theorem is not exhausted, since every upper bound on the semigroup can be used to improve the function m in (4.3).

5 Weyl asymptotics and random perturbations in a one-dimensional semi-classical case

We consider a simple model operator in dimension 1 and show how random perturbations give rise to Weyl asymptotics in the interior of the range of p. We follow rather closely the work of Hager [28] with some inputs also from Bordeaux Montrieux [3] and Hager–Sj [29]. Some of the general ideas appear perhaps more clearly in this special situation. For more general results, see the lecture notes [63].

Let $P = hD_x + g(x)$, $g \in C^\infty(S^1)$ with symbol $p(x,\xi) = \xi + g(x)$, and assume that $\Im g$ has precisely two critical points; a unique maximum and a unique minimum.

Let $\Omega \Subset \{z \in \mathbf{C};\, \min \Im g < \Im z < \max \Im g\}$ be open. Put

$$\begin{aligned} P_\delta = P_{\delta,\omega} &= hD_x + g(x) + \delta Q_\omega, \\ Q_\omega u(x) &= \sum_{|k|,|\ell| \leq \frac{C_1}{h}} \alpha_{j,k}(\omega)(u|e^k)e^\ell(x), \end{aligned} \tag{5.1}$$

where $C_1 > 0$ is sufficiently large, $e^k(x) = (2\pi)^{-1/2}e^{ikx}$, $k \in \mathbf{Z}$, and $\alpha_{j,k} \sim \mathcal{N}(0,1)$ are independent complex Gaussian random variables, centered with variance 1 (cf (5.12 below). Q_ω is compact, so P_δ has discrete spectrum. Let $\Gamma \Subset \Omega$ have smooth boundary.

Theorem 5.1 *Let $\kappa > 5/2$ and let $\epsilon_0 > 0$ be sufficiently small. Let $\delta = \delta(h)$ satisfy $e^{-\epsilon_0/h} \ll \delta \ll h^\kappa$ and put $\epsilon = \epsilon(h) = h\ln(1/\delta)$. Then for $h > 0$*

small enough, we have with probability $\geq 1 - \mathcal{O}(\frac{\delta^2}{\sqrt{\epsilon}h^5})$ *that the number of eigenvalues of* P_δ *in* Γ *satisfies*

$$|\#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{2\pi h}\mathrm{vol}(p^{-1}(\Gamma))| \leq \mathrm{Const.}\, \frac{\sqrt{\epsilon}}{h}. \tag{5.2}$$

If instead, we let Γ vary in a set of subsets that satisfy the assumptions uniformly, then with probability $\geq 1 - \mathcal{O}(\frac{\delta^2}{\epsilon h^5})$ we have (5.2) uniformly for all Γ in that subset. The remainder of the section is devoted to the (outline of) the proof of this result.

Remark 5.2 *The estimate on the probability in Theorem 5.1 is quite rough and can be improved by adapting the arguments in [29] where the corresponding multidimensional result was obtained.*

5.1 Preparations for the unperturbed operator

For $z \in \Omega$, let $x_+(z), x_-(z) \in S^1$ be the solutions of the equation $\Im g(x) = \Im z$, with $\pm \Im g'(x_\pm) < 0$, define $\xi_\pm(z)$ by $\xi_\pm + \Re g(x_\pm) = \Re z$. Then, with $\rho_\pm = (x_\pm, \xi_\pm)$, we have

$$p(\rho_\pm) = z, \quad \pm\frac{1}{i}\{p, \overline{p}\}(\rho_\pm) > 0.$$

We introduce quasimodes of the form

$$e_{\mathrm{wkb}}(x) = h^{-1/4}a(h)\chi(x - x_+(z))e^{\frac{i}{h}\phi_+(x)},$$

where $a(h) \sim a_0 + ha_1 + ..,\ a_0 \neq 0,\ \phi_+(x) = \int_{x_+(z)}^x (z - g(y))dy,\ \chi \in C_0^\infty(\mathrm{neigh}\,(0, \mathbf{R}))$ and $\chi(x) = 1$ in a neighborhood of 0. We can choose a depending smoothly on z such that all derivatives with respect to $z, \overline{z}$ are bounded when $h \to 0$ and $\|e_{\mathrm{wkb}}\| = 1$ where we take the L^2 norm over $]x_-(z), x_+(z) + 2\pi[$. We can assume that e_{wkb} is normalized in L^2 and

$$(P - z)e_{\mathrm{wkb}} = \mathcal{O}(e^{-\frac{1}{Ch}}).$$

Define z-dependent elliptic self-adjoint operators

$$Q = (P - z)^*(P - z),\ \widetilde{Q} = (P - z)(P - z)^* : L^2(S^1) \to L^2(S^1),$$

with domain $\mathcal{D}(Q)$, $\mathcal{D}(\widetilde{Q}) = H^2(S^1)$. They have discrete spectrum $\subset [0, +\infty[$. Using that $P - z$ is Fredholm of index zero, we see that $\dim \mathcal{N}(Q) = \dim \mathcal{N}(\widetilde{Q})$. If $\mu \neq 0$ is an eigenvalue of Q, with the corresponding eigenfunction $e \in C^\infty$, then $f := (P - z)e$ is an eigenfunction for $\widetilde{Q}$ with the same eigenvalue μ. Pursuing this observation, we see that

$$\sigma(Q) = \sigma(\widetilde{Q}) = \{t_0^2, t_1^2, ...\},\ 0 \leq t_j \nearrow +\infty.$$

Proposition 5.3 *There exists a constant $C > 0$ such that $t_0^2 = \mathcal{O}(e^{-1/(Ch)})$, $t_1^2 - t_0^2 \geq h/C$ for $h > 0$ small enough.*

Proof. We have $Qe_{\rm wkb} = r$, $\|r\| = \mathcal{O}(e^{-1/Ch})$ and since Q is self adjoint we deduce that t_0^2 is exponentially small. If e_0 denotes the corresponding normalized eigenfunction, we see that $(P-z)e_0 =: v$ with $\|v\|$ exponentially small. Considering this ODE on $]x_-(z) - 2\pi, x_-(z)[$, we get

$$e_0(x) = Ch^{-\frac{1}{4}} a(h) e^{\frac{i}{h}\phi_+(x)} + Fv(x),$$

$$Fv(x) = \frac{i}{h} \int_{x_+}^{x} e^{\frac{i}{h}(\phi_+(x) - \phi_+(y))} v(y) dy,$$

where $\phi_+(x) = \int_{x_+}^x (z - g(y))dy$. We observe that $\Im(\phi_+(x) - \phi_+(y)) \geq 0$ on the domain of integration. By applying the Shur lemma, Hager showed that $\|F\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(h^{-1/2})$. Hence for our particular v, we see that Fv is exponentially decaying in L^2. Recalling the form of $e_{\rm wkb}(x)$ we conclude that $\|e_0 - e_{\rm wkb}\|$ is exponentially small.

To show that $t_1^2 - t_0^2 \geq h/C$, it suffices to show that $(Qu|u) \geq \frac{h}{C}\|u\|^2$ when $u \perp e_0$ or in other words, that

$$\|u\| \leq \sqrt{\frac{C}{h}} \|(P-z)u\|. \tag{5.3}$$

If $v := (P-z)u$, we again have on $]x_-(z) - 2\pi, x_-(z)[$:

$$u = Ch^{-\frac{1}{4}} a(h) e^{\frac{i}{h}\phi_+(x)} + Fv$$

for some constant C and the orthogonality requirement on u implies that

$$0 = (1 + \mathcal{O}(h^\infty))C + (Fv|e_0),$$

where $(Fv|e_0) = \mathcal{O}(h^{-\frac{1}{2}})\|v\|$, so $C = \mathcal{O}(h^{-1/2})\|v\|$ and we get the desired estimate on $\|u\|$. $\Box$

5.2 Grushin (Shur, Feschbach, bifurcation) approach

Let f_0 be the normalized eigenfunction such that $\widetilde{Q} f_0 = t_0^2 f_0$. As observed prior to Proposition 5.3, we get

$$(P-z)e_0 = \alpha_0 f_0, \ (P-z)^* f_0 = \beta_0 e_0, \quad \alpha_0 \beta_0 = t_0^2,$$

and combining this with $((P-z)e_0|f_0) = (e_0|(P-z)^* f_0)$, we see that $\alpha_0 = \overline{\beta}_0$. Define $R_+ : L^2(S^1) \to \mathbf{C}$, $R_- = \mathbf{C} \to L^2(S^1)$ by

$$R_+ u = (u|e_0), \quad R_- u_- = u_- f_0.$$

Then

$$\mathcal{P}(z) := \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1 \times \mathbf{C} \to L^2 \times \mathbf{C}$$

is bijective with the bounded inverse

$$\mathcal{E}(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Here $E = \mathcal{O}(h^{-1/2})$ in $L^2 \to L^2$ is basically the inverse of $P - z$ from $(f_0)^\perp$ to $(e_0)^\perp$, $E_- v = (v|f_0)$, $E_+ v_+ = v_+ e_0$, $E_{-+} = \mathcal{O}(e^{-1/(Ch)})$. It is a general feature of such auxiliary (Grushin) operators that

$$z \in \sigma(P) \Leftrightarrow E_{-+}(z) = 0.$$

5.3 d-bar equation for E_{-+}

Proposition 5.4 *We have*

$$\partial_{\overline{z}} E_{-+}(z) + f(z) E_{-+}(z) = 0, \tag{5.4}$$

where

$$f(z) = f_+(z) + f_-(z), \quad f_+(z) = (\partial_{\overline{z}} R_+) E_+, \;\; f_-(z) = E_- \partial_{\overline{z}} R_-. \tag{5.5}$$

Thus,

$$\partial_{\overline{z}}(e^{F(z)} E_{-+}(z)) = 0 \text{ if } \partial_{\overline{z}} F(z) = f(z). \tag{5.6}$$

Moreover,

$$\Re \Delta F(z) = \Re 4 \partial_z f = \frac{2}{h}\Big(\frac{1}{\frac{1}{i}\{p, \overline{p}\}(\rho_+)} - \frac{1}{\frac{1}{i}\{p, \overline{p}\}(\rho_-)}\Big) + \mathcal{O}(1). \tag{5.7}$$

Proof. (5.4), (5.5) follow from the general formula for the differentiation of the inverse of an operator, here:

$$\partial_{\overline{z}} \mathcal{E} + \mathcal{E}(\partial_{\overline{z}} \mathcal{P}) \mathcal{E} = 0.$$

Let $\Pi(z)$ be the spectral projection of Q: $L^2 \to \mathbf{C} e_0$. It is easy to see that the various z and $\overline{z}$ derivatives of $e_{\rm wkb}$ and $\Pi(z)$ have at most temperate growth

in $1/h$ and since e_0 is the normalization of $\Pi(z)e_{\rm wkb}$ we get the same fact for e_0 and hence for $e_0 - e_{\rm wkb}$. This quantity is also exponentially small in L^2 and by elementary interpolation estimates for the successive derivatives in $z, \overline{z}$ we get the same conclusion for the higher z-gradients of $e_0 - e_{\rm wkb}$.
It follows that

$$f_+(z) = (e_0(z)|\partial_z e_0(z)) = (e_{\rm wkb}(z)|\partial_z e_{\rm wkb}(z)) + \mathcal{O}(e^{-\frac{1}{Ch}}),$$

and the various $z, \overline{z}$-derivatives of the remainder are also exponentially decaying.

Using that $e_{\rm wkb}$ behaves like a Gaussian, peaked at the point $x_+(\zeta)$, we can apply a variant of the method of stationary phase to get

$$(e_{\rm wkb}|\partial_z e_{\rm wkb}) = -\frac{i}{h}\overline{(\partial_z\phi_+)(x_+(z), z)} + \mathcal{O}(1), \tag{5.8}$$

where the remainder remains bounded after taking $z, \overline{z}$ derivatives.
Using that $\phi_+(x_+(z), z) = 0$, $(\phi_+)'_x(x_+(z), z) = \xi_+(z)$, we get after applying ∂_z to the first of these relations, that

$$(\partial_z\phi_+)(x_+(z), z) = -\xi_+(z)\partial_z x_+(z).$$

On the other hand, if we apply ∂_z and $\partial_{\overline{z}}$ to the equation, $p(x_+(z), \xi_+(z)) = z$, we get

$$\begin{cases} p'_x\partial_z x_+ + p'_\xi\partial_z\xi_+ = 1 \\ p'_x\partial_{\overline{z}} x_+ + p'_\xi\partial_{\overline{z}}\xi_+ = 0 \end{cases} \tag{5.9}$$

and using that $x_+(z)$ and $\xi_+(z)$ are real valued we get

$$\begin{cases} p'_x\partial_z x_+ + p'_\xi\partial_z x_+ = 1 \\ \overline{p}'_x\partial_z x_+ + \overline{p}'_\xi\partial_z\xi_+ = 0 \end{cases} \tag{5.10}$$

which can be solved and we get

$$\partial_{\overline{z}} x_+ = \frac{p'_\xi}{\{p, \overline{p}\}}(\rho_+), \quad \partial_{\overline{z}}\xi_+ = \frac{-p'_x}{\{p, \overline{p}\}}(\rho_+).$$

Plugging this into (5.8), applying ∂_z and taking real parts, we get the second (non-trivial) identity in (5.7) for the contribution from f_+. The one from f_- can be treated similarly. □

Using the expressions for the z-derivatives of x_+, ξ_+ and the analogous ones for x_-, ξ_-, we have the following easy result relating (5.7) to the symplectic volume:

Proposition 5.5 *Writing $z = x + iy$, we have:*

$$d\xi_+(z) \wedge dx_+(z) = \frac{2}{\frac{1}{i}\{p,\overline{p}\}(\rho_+)} dy \wedge dx,$$

$$-d\xi_-(z) \wedge dx_-(z) = -\frac{2}{\frac{1}{i}\{p,\overline{p}\}(\rho_-)} dy \wedge dx,$$

so by (5.7),

$$\Re \Delta F(z) dy \wedge dx = \frac{1}{h}(d\xi_+ \wedge dx_+ - d\xi_- \wedge dx_-) + \mathcal{O}(1). \tag{5.11}$$

5.4 Adding the random perturbation

Let $X \sim \mathcal{N}_{\mathbf{C}}(0, \sigma^2)$ be a complex Gaussian random variable, meaning that X has the probability distribution

$$X_*(P(d\omega)) = \frac{1}{\pi\sigma^2} e^{-\frac{|X|^2}{\sigma^2}} d(\Re X) d(\Im X). \tag{5.12}$$

Here $\sigma > 0$. For $t < 1/\sigma^2$, we have the expectation value

$$E(e^{t|X|^2}) = \frac{1}{1 - \sigma^2 t}. \tag{5.13}$$

Bordeaux Montrieux [3] observed that we have the following possibly classical result (improving a similar statement in [29]).

Proposition 5.6 *There exists $C_0 > 0$ such that the following holds: Let $X_j \sim \mathcal{N}_{\mathbf{C}}(0, \sigma_j^2)$, $1 \leq j \leq N < \infty$ be independent complex Gaussian random variables. Put $s_1 = \max \sigma_j^2$. Then for every $x > 0$, we have*

$$P(\sum_1^N |X_j|^2 \geq x) \leq \exp(\frac{C_0}{2s_1} \sum_1^N \sigma_j^2 - \frac{x}{2s_1}).$$

Proof. For $t \leq 1/(2s_1)$, we have

$$P(\sum |X_j|^2 \geq x) \leq E(e^{t(\sum |X_j|^2 - x)}) = e^{-tx} \prod_1^N E(e^{t|X_j|^2})$$

$$= \exp(\sum_1^N \ln \frac{1}{1 - \sigma_j^2 t} - tx) \leq \exp(C_0 \sum \sigma_j^2 t - tx).$$

It then suffices to take $t = (2s_1)^{-1}$. $\square$

Recall that

$$Q_\omega u(x) = \sum_{|k|,|j|\le C_1/h} \alpha_{j,k}(\omega)(u|e^k)e^j(x),\ \ e^k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}. \tag{5.14}$$

Since the Hilbert-Schmidt norm of Q_ω is given by $\|Q_\omega\|_{\mathrm{HS}}^2 = \sum |\alpha_{j,k}(\omega)|^2$, we get from the preceding proposition:

Proposition 5.7 *If $C > 0$ is large enough, then*

$$\|Q_\omega\|_{\mathrm{HS}} \le \frac{C}{h} \text{ with probability } \ge 1 - e^{-\frac{1}{Ch^2}}. \tag{5.15}$$

Now, we work under the assumption that $\|Q_\omega\|_{\mathrm{HS}} \le C/h$ and recall that $\|Q_\omega\|_{\mathcal{L}(L^2,L^2)} \le \|Q_\omega\|_{\mathrm{HS}}$. Assume that

$$\delta \ll h^{3/2}, \tag{5.16}$$

so that $\|\delta Q_\omega\| \ll h^{1/2}$. Then, by simple perturbation theory we see that

$$\mathcal{P}_\delta(z) = \begin{pmatrix} P_\delta - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1 \times \mathbf{C} \to L^2 \times \mathbf{C}$$

is bijective with the bounded inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}$$

$$\begin{aligned} E^\delta &= E + \mathcal{O}(\frac{\delta}{h^2}) = \mathcal{O}(h^{-1/2}) \text{ in } \mathcal{L}(L^2, L^2) \\ E_+^\delta &= E_+ + \mathcal{O}(\frac{\delta}{h^{3/2}}) = \mathcal{O}(1) \text{ in } \mathcal{L}(\mathbf{C}, L^2) \\ E_-^\delta &= E_- + \mathcal{O}(\frac{\delta}{h^{3/2}}) = \mathcal{O}(1) \text{ in } \mathcal{L}(L^2, \mathbf{C}) \\ E_{-+}^\delta &= E_{-+} - \delta E_- Q E_+ + \mathcal{O}(\frac{\delta^2}{h^{5/2}}). \end{aligned} \tag{5.17}$$

As before the eigenvalues of P_δ are the zeros of E_{-+}^δ and we have the d-bar equation

$$\partial_{\overline{z}} E_{-+}^\delta + f^\delta(z) E_{-+}^\delta = 0,$$

$$f^\delta(z) = \partial_{\overline{z}} R_+ E_+^\delta + E_-^\delta \partial_{\overline{z}} R_- = f(z) + \mathcal{O}(\frac{1}{h}\frac{\delta}{h^{3/2}}).$$

We can solve $\partial_{\overline{z}} F^\delta = f^\delta$ (making $e^{F^\delta} E_{-+}^\delta$ holomorphic) with

$$F^\delta = F + \mathcal{O}(\frac{\delta}{h^{4/2}}) = F + \mathcal{O}(\frac{\delta}{h^{3/2}})\frac{1}{h}. \tag{5.18}$$

Proposition 5.8 *Assume that* $0 < t \ll 1$, $\delta \ll h^{3/2}$,

$$\delta t \gg e^{-\frac{1}{C_0 h}}, \quad t \gg \frac{\delta}{h^{5/2}}, \tag{5.19}$$

where $C_0 \gg 1$ *is fixed. Then with probability* $\geq 1 - e^{-\frac{1}{Ch^2}}$, *we have*

$$|E_{-+}^{\delta}(z)| \leq e^{-\frac{1}{Ch}} + \frac{C\delta}{h}, \ \forall z \in \Omega. \tag{5.20}$$

For every $z \in \Omega$, *we have with probability* $\geq 1 - \mathcal{O}(t^2) - e^{-\frac{1}{Ch^2}}$, *that*

$$|E_{-+}^{\delta}(z)| \geq \frac{t\delta}{C}, \tag{5.21}$$

We only give the main idea of the proof which is to notice that $E_- Q_\omega E_+$ can be written as a sum of independent Gaussian random variables and is therefore itself a Gaussian random variable. Applying the standard formula for the variance of such a sum we get for the variance:

$$\sigma^2 = \sum_{|k|,|j| \leq \frac{C_1}{h}} |\widehat{e}_0(j)|^2 |\widehat{f}_0(k)|^2, \tag{5.22}$$

where $\widehat{e}_0(j)$, $\widehat{f}_0(j)$ are the Fourier coefficients of e_0, f_0. Now we can show that the Fourier coefficients are $\mathcal{O}((h/|j|)^N)$ for every $N \geq 0$, when $h|j|$ is sufficiently large, so if we take C_1 (in the definition of Q_ω) large enough, we conclude that $\sigma^2 = 1 + \mathcal{O}(h^\infty)$.
The remainder of the proof then consists in showing that $|E_- Q_\omega E_+|$ is $\geq t$ with probability $\geq 1 - \mathcal{O}(t^2)$ and observing that when this happens, then the second term in the expression for E_{-+}^{δ} in (5.17) is dominant. □

Proposition 5.9 *Let* $\kappa > 5/2$ *and fix* $\epsilon_0 \in]0, 1[$ *sufficiently small. Let* $\delta = \delta(h)$ *satisfy* $e^{-\epsilon_0/h} \ll \delta \ll h^\kappa$, *and put* $\epsilon = \epsilon(h) = h \ln \frac{1}{\delta}$. *Then with probability* $\geq 1 - e^{-1/(Ch^2)}$ *we have* $|E_{-+}^{\delta}| \leq 1$ *for all* $z \in \Omega$.
For any $z \in \Omega$, *we have* $|E_{-+}^{\delta}| \geq e^{-C\epsilon/h}$ *with probability* $\geq 1 - \mathcal{O}(\delta^2/h^5)$.

This follows from Proposition 5.8 by choosing t such that

$$\max\left(\frac{1}{\delta} e^{-\frac{1}{C_0 h}}, \frac{\delta}{h^{5/2}}, C\delta^{C-1}\right) \ll t \leq \mathcal{O}\left(\frac{\delta}{h^{5/2}}\right),$$

which is possible to do since

$$\frac{1}{\delta} e^{-\frac{1}{C_0 h}}, \ C\delta^{C-1} \ll \frac{\delta}{h^{5/2}}.$$

Under the same assumptions, we also have

$$|F_\delta - F| \le \mathcal{O}(\frac{\delta}{h^{3/2}})\frac{1}{h} \le \mathcal{O}(\epsilon)\frac{1}{h}.$$

Thus for the holomorphic function $u(z) = e^{F_\delta(z)}E_{-+}^\delta(z)$ we have

- With probability $\ge 1 - e^{-1/(Ch^2)}$ we have $|u(z)| \le \exp(\Re F(z) + C\epsilon/h)$ for all $z \in \Omega$.
- For every $z \in \Omega$, we have $|u(z)| \ge \exp(\Re F(z) - C\epsilon/h)$ with probability $\ge 1 - \mathcal{O}(\delta^2/h^5)$.

Theorem 5.1 on the Weyl asymptotics of small random perturbations of the operator $P = hD + g(x)$ is now a consequence of the following result of M. Hager, that we apply with $\phi = h\Re F$

Proposition 5.10 *Let $\Gamma \Subset \mathbf{C}$ have smooth boundary and let ϕ be a real valued C^2-function defined in a fixed neighborhood of $\overline{\Gamma}$. Let $z \mapsto u(z;h)$ be a family of holomorphic functions defined in a fixed neighborhood of $\overline{\Gamma}$, and let $0 < \epsilon = \epsilon(h) \ll 1$. Assume*

- $|u(z;h)| \le \exp(\frac{1}{h}(\phi(z) + \epsilon))$ *for all z in a fixed neighborhood of $\partial\Gamma$.*
- *There exist $z_1, ..., z_N$ depending on h, with $N = N(h) \asymp \epsilon^{-1/2}$ such that $\partial\Gamma \subset \cup_1^N D(z_k, \sqrt{\epsilon})$ such that $|u(z_k;h)| \ge \exp(\frac{1}{h}(\phi(z_k) - \epsilon))$, $1 \le k \le N(h)$.*

Then, the number of zeros of u in Γ satisfies

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\int_\Gamma \Delta\phi(z)dxdy| \le C\frac{\sqrt{\epsilon}}{h}.$$

There is a more general theorem that will probably be useful and which allows a less regular exponent ϕ and h-dependent Lipschitz domains. See [65].

5.5 Proof of Proposition 5.10, an outline

Define $\phi_j(z)$ by $i\phi_j(z) = \phi(z_j) + 2\partial_z\phi(z_j)(z - z_j)$. Then

$$\begin{aligned} \phi(z) &= \Re(i\phi_j(z)) + R_j(z),\ R_j(z) = \mathcal{O}((z-z_j)^2) \\ \phi_j'(z) &= \frac{2}{i}\partial_z\phi(z) + \mathcal{O}((z-z_j)). \end{aligned}$$

Consider the holomorphic function

$$v_j(z;h) = u(z;h)e^{-\frac{i}{h}\phi_j(z)}.$$

Then $|v_j(z;h)| \leq e^{\frac{1}{h}(\phi(z)-\Re i\phi_j(z))} = e^{\frac{1}{h}R_j} \leq e^{\frac{C\epsilon}{h}}$, when $z - z_j = \mathcal{O}(\sqrt{\epsilon})$, while

$$|v_j(z_j;h)| \geq e^{-\frac{C\epsilon}{h}}.$$

In a $\sqrt{\epsilon}$-neighborhood of z_j we put $v = v_j$ and make the change of variables $w = (z - z_j)/\sqrt{\epsilon}$, $\widetilde{v}(w) = v(z)$, so that

$$|\widetilde{v}(w)| \leq e^{C\epsilon/h} \text{ on } D(0,2),\ \ |\widetilde{v}(0)| \geq e^{-C\epsilon/h}.$$

Using Jensen's formula we see that the number of zeros $w_1, ..., w_N$ of $\widetilde{v}$ in $D(0, 3/2)$ (repeated with their multiplicity) is $\mathcal{O}(\epsilon/h)$. Factorize:

$$\widetilde{v}(w) = e^{g(w)} \prod_1^N (w - w_k).$$

Using the maximum principle and a suitably chosen disc of radius between 4/3 and 3/2, and then also Harnack's inequality we can follow a standard procedure to show that

$$\Re g(w),\ g'(w) \ = \mathcal{O}(\epsilon/h) \text{ in } D(0, 6/5).$$

Using finally that $\partial\Gamma$ is covered by the discs $D(z_j, \sqrt{\epsilon})$ and using the above representation of u in each disc, we can show that the number of zeros of $u(\cdot;h)$ in Γ is equal to

$$\begin{aligned}\Re\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{u'(z)}{u(z)}dz &= \Re\frac{1}{2\pi h}\int_{\partial\Gamma}\frac{2}{i}\partial_z\phi(z)dz + \mathcal{O}(\frac{\sqrt{\epsilon}}{h}) \\ &= \frac{1}{2\pi h}\int \Delta\phi(x)dxdy + \mathcal{O}(\frac{\sqrt{\epsilon}}{h}). \qquad \square\end{aligned}$$

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