Lectures on resonances

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1 Introduction

This text is the written version of lectures on resonances given at the Mathematics department of the Gothenburg University and Chalmers in the Spring semesters of 2000 to 2002. Some of the first chapters were also given in a graduate course at Ecole Polytechnique. Despite the somewhat preliminary form of these notes we hope they can be of some use as an introduction to the subject, and some guidance for a deeper study (necessarily involving the reading of research papers). We hope to improve and complete the text gradually.

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2 Compactly supported perturbations of $-\Delta_{\mathbf{R}^n}$, definition of resonances and their multiplicity.

2.1 The free resolvent.

We start by recalling some facts about the "free Laplacian" $P_0 = -\Delta = \sum_1^n -\frac{\partial^2}{\partial x_j^2}$ and its resolvent. First of all, $-\Delta$ is an unbounded selfadjoint operator $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ with domain $H^2(\mathbf{R}^n)$ (the standard Sobolev space). By Fourier transform it is unitarily equivalent to the (unbounded) multiplication operator $L^2(\mathbf{R}^2) \ni \widehat{u}(\xi) \mapsto \xi^2\widehat{u}(\xi) \in L^2(\mathbf{R}^n)$. The spectrum of $-\Delta$ is equal to $[0, +\infty[$ (and it is purely absolutely continuous).

For $\operatorname{Im}\lambda > 0$, we know that $\lambda^2 \notin [0, +\infty[$, so $P_0 - \lambda^2 : H^2(\mathbf{R}^2) \to L^2(\mathbf{R}^n)$ is bijective with inverse $R_0(\lambda)$, which is a convolution operator with locally integrable kernel

$$R_0(\lambda; x-y) = \frac{i}{4}\left(\frac{1}{2\pi|x-y|}\right)^{\frac{n-2}{2}}\left(u^{\frac{n-2}{2}}H^{(1)}_{\frac{n-2}{2}}(u)\right)_{|_{u=\lambda|x-y|}}, \tag{2.1}$$

where $H^{(1)}_{\frac{n-2}{2}}$ is a Hankel function. We can deduce (see Vainberg [91]) that

$$R_0(\lambda) = \begin{cases} E(\lambda) + \lambda^{n-2}E_1(\lambda), & n \text{ odd} \\ F(\lambda) + F_1(\lambda)\lambda^{n-2}\log\lambda, & n \text{ even}, \end{cases} \tag{2.2}$$

where $E(\lambda)$, $E_1(\lambda)$, $F(\lambda)$, $F_1(\lambda)$ are entire functions with values in the space of convolution operators and in the space of classical pseudodifferential operators of order -2: $H_{\mathrm{comp}}(\mathbf{R}^n) \to H^2_{\mathrm{loc}}(\mathbf{R}^n)$.

If $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp}\chi_1 \cap \chi_2 = \emptyset$, then $\chi_1(\text{any of } E, E_1, F, F_1)\chi_2$ is an entire function in λ with values in $\mathcal{L}(H^{-s}, H^s)$, for every $s > 0$.

Further for n even:

$$R_0(e^{im\pi}z) = R_0(z) - \left(\frac{im}{4\pi}\right)(-1)^{\frac{n-2}{2}(m+1)}T(z) \tag{2.3}$$

for $0 \le \arg \le \pi$ $(z \neq 0,)$ $m \in \mathbf{Z}$, and $e^{im\pi}z$ is viewed as a point on the logarithmic covering space of $\mathbf{C}\setminus\{0\}$. Here $T(z)$ is the convolution operator with kernel

$$T(z, x-y) = \left(\frac{z}{2\pi}\right)^{n-2}\int_{S^{n-1}} e^{iz(x-y)w}L(dw), \tag{2.4}$$

where $L(dw)$ denotes the standard Lebesgue measure on S^{n-1}.

For n odd we have

$$R_0(z) - R_0(-z) = T(z), \tag{2.5}$$

where

$$T(z, x-y) = \frac{i}{2} \frac{z^{n-2}}{(2\pi)^{n-1}} \int_{S^{n-1}} e^{iz(x-y)w} L(dw). \tag{2.6}$$

The basic facts to remember are that $R_0(\lambda)$, initially defined for $\operatorname{Im} \lambda > 0$ and viewed as a convolution operator $H^0_{\mathrm{comp}}(\mathbf{R}^n) \to H^2_{\mathrm{loc}}(\mathbf{R}^n)$, can be extended holomorphically to:
– $\lambda \in \mathbf{C}$, when n is odd ≥ 3,
– $\lambda \in \mathbf{C} \setminus \{0\}$, when $n = 1$, and the extension has a simple pole at $\lambda = 0$,
– to the logarithmic (universal) covering space of $\mathbf{C} \setminus \{0\}$, when n is even.

These properties have an interesting relation to the strong Huygens principle for the wave equation. It is well known that the Cauchy problem in $\mathbf{R} \times \mathbf{R}^n$:

$$(\partial_t^2 - \Delta) U_0(t) = 0, \; U_0(0) = 0, \; U_0'(0) = \delta, \tag{2.7}$$

has a unique solution $U_0 \in C^\infty(\mathbf{R}_t; \mathcal{D}'(\mathbf{R}^n_x))$ and that

$$\operatorname{supp} U_0 \subset \{(t,x); \, |x| \leq |t|\}.$$

This is the Huygens principle which is valid more generally for wave equations with variable coefficients and more general hyperbolic equations, and which says that the support of a solution cannot travel faster than the natural wave propagation speed (which is one in our special case). When n is odd and ≥ 3, we have the strong Huygens principle:

$$\operatorname{supp} U_0 = \{(t,x); |x| = |t|\}$$

This principle has the important practical consequence that we can communicate and send intelligible signals using sound waves and electromagnetic waves, since we are lucky enough to live in a three dimensional world.

By Duhamel's principle, we have the forward fundamental solution of the wave operator $E_0 \in \mathcal{S}'(\mathbf{R} \times \mathbf{R}^n)$ with

$$(\partial_t^2 - \Delta) E_0 = \delta_{\mathbf{R}^{n+1}}, \;\; E_0 \subset \{(t,x); \, t \geq |x|\}, \tag{2.8}$$

given by

$$E_0 = H(t) U_0(t), \tag{2.9}$$

where $H(t) = 1_{[0,+\infty[}(t)$ is the Heaviside function. Viewing U_0 as a convolution operator, solving $(\partial^2 - \Delta)U_0(t) = 0$, $U_0(0) = 0$, $U_0'(0) = 1$ (1 indicating the identity operator) we can write

$$U_0(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}. \tag{2.10}$$

Without any general spectral theory, (2.10) simply says that for the Fourier transform with respect to x, we have

$$\widehat{U_0(t)}(\xi) = \frac{\sin t|\xi|}{|\xi|},$$

a fact that can be easily verified by taking the Fourier transform with respect to x of (2.8).

On the other hand, for $\operatorname{Im}\lambda > 0$, it is easy to see that

$$R_0(\lambda) = \int_0^\infty e^{it\lambda}U_0(t)dt. \tag{2.11}$$

This will be verified in a more general situation in Chapter 3. The same formula holds on the level of convolution kernels. If n is odd ≥ 3, we see that $U_0(t, x) = 0$ for $t > |x|$ by the strong Huygens principle, so in a region $|x| < R$ we have

$$R_0(\lambda, x) = \int_0^{R+1} e^{it\lambda}U_0(t, x)dt, \tag{2.12}$$

which can be extended holomorphically in λ to all of $\mathbf{C}$.

Using (2.11), we can also derive explicit formulas for R_0, from corresponding explicit formulas for the fundamental solution. In odd dimensions the computations are easier. It is known (see volume 1 of [42]) that we have for $t > 0$:

$$\langle U_0(t), \chi\rangle = \frac{1}{4\pi^{\frac{n-1}{2}}}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\omega|=1}\chi(t\omega)L(d\omega), \tag{2.13}$$

where $L(d\omega)$ denotes the Lebesgue measure, and n is odd and ≥ 3. Take $n = 3$ for simplicity, so that

$$\langle U_0(t), \chi\rangle = \frac{1}{4\pi}t\int_{|\omega|=1}\chi(t\omega)L(d\omega). \tag{2.14}$$

Then

$$\langle R_0(\lambda), \chi\rangle = \frac{1}{4\pi}\int_{|\omega|=1}\int_0^\infty e^{it\lambda}\chi(t\omega)tdtL(d\omega). \tag{2.15}$$

With $x = t\omega$, $t = |x|$, $|\omega| = 1$, we have $t^2 dt d\omega = dx$, so

$$\langle R_0(\lambda), \chi\rangle = \frac{1}{4\pi}\int_{\mathbf{R}^n} \frac{1}{|x|} e^{i|x|\lambda}\chi(x)dx, \tag{2.16}$$

or in other terms,

$$R_0(\lambda, x) = \frac{e^{i|x|\lambda}}{4\pi|x|}. \tag{2.17}$$

For higher odd values of n one gets a similar formula with the same exponential factor and a more complicated polynomial expression in front. For even n the expression is even more complicated and contains special functions. For $n = 1$ the computation is of course easy and is left as an exercise. In all cases, $R_0(\lambda, x)$ is smooth and even analytic for $x \neq 0$, locally integrable in x, and exponentially decaying (for λ in the upper halfplane) when $x \to \infty$. These properties can also be deduced by means of Fourier inversion.

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Here is a more general way to understand the link between the dimension and the extendability properties. Starting from the physical halfplane $\operatorname{Im}\lambda > 0$, we use Fourier inversion to write

$$R_0(\lambda, x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\frac{1}{\xi^2 - \lambda^2}d\xi. \tag{2.18}$$

To avoid uninteresting convergence problems, we take $\widehat{\phi} \in C_0^\infty(\mathbf{R}^n)$, so that $\phi = \mathcal{F}^{-1}\widehat{\phi}$ is of class $\mathcal{S}$ on $\mathbf{R}^n$ and extend to an entire function on $\mathbf{C}^n$. In the sense of distributions, we get

$$\langle R_0(\lambda), \widehat{\phi}\rangle = \int \frac{1}{(\xi^2 - \lambda^2)}\phi(\xi)d\xi. \tag{2.19}$$

Now pass to polar coordinates, but let r vary in $\mathbf{R}$ rather than in $[0, \infty[$, so that we get the double covering

$$\mathbf{R} \times S^{n-1} \ni (r, \omega) \mapsto r\omega \in \mathbf{R}^n. \tag{2.20}$$

Then the pull back of $d\xi$ becomes

$$|r|^{n-1}drdL(\omega) = \begin{cases} r^{n-1}drL(d\omega) \text{ for } n \text{ odd.} \\ (\operatorname{sgn} r)r^{n-1}drL(d\omega) \text{ for } n \text{ even.} \end{cases}$$

We can then write

$$\langle R_0(\lambda), \widehat{\phi}\rangle = \frac{1}{2}\int_{S^{n-1}}\int_\gamma \frac{\phi(r\omega)}{r^2-\lambda^2} r^{n-1} dr L(d\omega), \tag{2.21}$$

where γ is the positively oriented real axis when n is odd and the union of $[0,-\infty[$ and $[0,+\infty[$, both oriented from 0 to ∞, when n is even. Let λ start in the upper half plane and turn once around 0 in the clockwise direction. [[[[Figur!]]]] From this simple argument we see that $R_0(\lambda)$ extends to a holomorphic function on $\mathbf{C}\setminus\{0\}$ when n is odd, while we get a log type singularity when n is even. From this one can verify or at least understand qualitatively the earlier statements involving $T(\lambda)$.

2.2 Definition of resonances in the case of $P=-\Delta+V$, when V is bounded and compactly supported.

Let $V\in L^\infty_{\mathrm{comp}}(\mathbf{R}^n;\mathbf{R})$. Then $P:=P_0+V$ is an unbounded selfadjoint operator in $L^2(\mathbf{R}^n)$ with domain H^2. $[0,+\infty[$ is the essential spectrum, so in $]-\infty,0[$ there can only be isolated eigenvalues of finite multiplicity. We then know that the resolvent $(P-\lambda^2)^{-1}$ exists as a bounded operator $H^0\to H^2$, when $\operatorname{Im}\lambda>0$, $\lambda\neq i\mu_j$, where $-\mu_j^2$ are the negative eigenvalues. We want to find a meromorphic extension of the resolvent as a continuous operator $L^2_{\mathrm{comp}}(\mathbf{R}^n)\to H^2_{\mathrm{comp}}(\mathbf{R}^n)$ across $\mathbf{R}\setminus\{0\}$. Write

$$P-\lambda^2=(P_0-\lambda^2)+V=(1+VR_0(\lambda))(P_0-\lambda^2). \tag{2.22}$$

For $|\lambda|\gg 1$, $\arg\lambda=\frac{\pi}{4}$, it is clear that $(1+VR_0(\lambda))^{-1}$ exists. Moreover $VR_0(\lambda):L^2\to L^2$ is compact and depends holomorphically on λ for $\operatorname{Im}\lambda>0$. We conclude by analytic Fredholm theory (see Chapter 5) that $(1+VR_0(\lambda))^{-1}$ exists except for λ in some discrete set, where the inverse has poles of finite multiplicity, with finite rank coefficients in the singular terms of the Laurent expansions. (Actually these poles are precisely the $i\mu_j$, because (2.22) shows that $P-\lambda^2$ is bijective $H^2\to H^0$ precisely when $1+VR_0(\lambda)$ is bijective.)

After crossing $\mathbf{R}\setminus\{0\}$, VR_0 is in general no more a bounded operator in L^2, but we can still try to solve

$$(1+VR_0(\lambda))u=v \tag{2.23}$$

with u, v of compact support. Notice that the support of such a solution u will be contained in the union of the supports of v and V. Let $1_{\mathrm{supp}\, V} \prec \chi \in C_0^\infty$ where we write $\chi_1 \prec \chi_2$ for two functions χ_1, χ_2 if the support of χ_1 is contained in the interior of the support of $1 - \chi_2$. Then $1 + VR_0(\lambda)\chi$ is a holomorphic family of Fredholm operators (see Chapter 5) and we shall see that in order to solve (2.23) it suffices to invert $1 + VR_0(\lambda)\chi$.

Indeed, we have

$$1 + VR_0(\lambda) = (1 + VR_0(\lambda)(1-\chi))(1 + VR_0(\lambda)\chi), \tag{2.24}$$

$$1 = (1 + VR_0(\lambda)(1-\chi))(1 - VR_0(1-\chi)), \tag{2.25}$$

so

$$(1 + VR_0(\lambda)(1-\chi))^{-1} = 1 - VR_0(\lambda)(1-\chi) \text{ on } L^2_{\mathrm{comp}}. \tag{2.26}$$

Hence $(1+VR_0(\lambda)) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}$ is invertible precisely when $1+VR_0(\lambda)\chi$ is, and

$$(1 + VR_0(\lambda))^{-1} = (1 + VR_0(\lambda)\chi)^{-1}(1 - VR_0(\lambda)(1-\chi)) : \, L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}. \tag{2.27}$$

Inverting (2.22), we get

$$(P - \lambda^2)^{-1} = R_0(\lambda)(1 + VR_0(\lambda))^{-1}, \tag{2.28}$$

and by analytic Fredholm theory (see chapter 5), we know that $(1+VR_0(\lambda)(1-\chi))^{-1}$ has a meromorphic extension in λ to the complex plane when $n \geq 3$ is odd and to other sets already described in the other cases, as a function with values in the space $\mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$ of bounded operators from L^2 to L^2. By a meromorphic operator valued function of $z \in \Omega$, with values in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_j$ are complex Hilbert spaces and Ω is an open set in the complex plane or in a covering surface over some open set of $\mathbf{C}$, we mean a function $A(z)$ which is holomorphic in z on $\Omega \setminus S$, where S is a discrete subset of Ω and such that if $z_0 \in S$, then for z in a neighborhood of z_0, we have the Laurent series representation

$$A(z) = \sum_1^N \frac{A_j}{(z - z_0)^j} + B(z), \tag{2.29}$$

where N is finite (so that we have poles of finite order), $B(z)$ a holomorphic operator valued function in a neighborhood of z_0, and A_j are bounded operators, independent of z and of finite rank. We leave as an exercise to verify that compositions and sums of meromorphic operator valaued functions are still meromorphic. Meromorphic operator valued functiosn are also conserved under changes of variables $z = \kappa(\lambda)$ if $\kappa : \Lambda \to \Omega$ is a biholomorphic map beteween two open sets.

Proposition 2.1 *$(P - \lambda^2)^{-1} : H^0_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ has a meromorphic extension from the open upper half plane to:*
$\mathbf{C} \setminus \{0\}$, when $n = 1$,
$\mathbf{C}$, when $n \geq 3$ is odd,
the logarithmic covering space $(\mathbf{C} \setminus \{0\})^*$ of $\mathbf{C} \setminus \{0\}$, when n is even.

By definition, the resonances or scattering poles of P are the poles of this extension, with the exception of the $i\mu_j$. Since H^0_{comp}, H^2_{loc} are not Hilbert or even Banach spaces, we have to extend the definition of meromorphic and holomorphic functions with values in operators between such spaces. Let Ω be as above. A holomorphic function with values in $\mathcal{L}(H^0_{\mathrm{comp}}, H^2_{\mathrm{loc}})$ is a function $A(z)$ with values in the space of linear operators $H^0_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$, such that $\chi_1 A(z) \chi_2$ is holomorphic for all $\chi_j \in C_0^\infty(\mathbf{R}^n)$. Correspondingly a meromorphic function is one which is holomorphic on $\Omega \setminus S$, where $S \subset \Omega$ is discrete, and such that if $z_0 \in S$, then near z_0, we have (2.29), now with $A_j : H^0_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ (continuous in the sense that $\chi_1 A_j \chi_2$ is bounded for all χ_j as above) of finite rank, and $B(z)$ holomorphic with values in $\mathcal{L}(H^0_{\mathrm{comp}}, H^2_{\mathrm{loc}})$ for z in a neighborhood of z_0.

2.3 Black box frame work and definition of resonances for general compactly supported perturbations of $-\Delta$.

The black box frame work (introduced by Sjöstrand and Zworski in [80]) permits to give a uniform treatment of all sorts of compactly supported (and later also long range) perturbations of $-\Delta$, like exterior obstacle problems, metric pertubations, Schrödinger operators, or a combination of such problems.

Let $\mathcal{H}$ be a complex Hilbert space with the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbf{R}^n \setminus B(0, R_0)), \tag{2.30}$$

where $R_0 > 0$ and we use the standard notation $B(x, r) = \{y \in \mathbf{R}^n;\ |y - x| < r\}$. We denote by

$$u \mapsto u_{|_{B(0,R_0)}} \text{ or } 1_{B(0,R_0}u$$

$$u \mapsto u_{|\mathbf{R}^n\setminus B(0,R_0)} \text{ or } 1_{\mathbf{R}^n\setminus B(0,R_0)}u,$$

the corresponding orthogonal projections. If χ is a bounded continuous function on $B(0,R_0)$; $\chi \in C_b(\mathbf{R}^n)$, and $\chi = C_0 = \text{Const.}$ on $B(0,R_0)$, then for $u \in \mathcal{H}$, we can define $\chi u \in \mathcal{H}$ by

$$\chi u = C_0 u + 1_{\mathbf{R}^n\setminus B(0,R_0)}(\chi - C_0)u. \tag{2.31}$$

Let

$$\begin{aligned} &\mathcal{H}_{\mathrm{comp}} = \{u \in \mathcal{H};\, u_{|\mathbf{R}^n\setminus B(0,R_0)} \text{ has bounded support}\},\\ &\mathcal{H}_{\mathrm{loc}} = \mathcal{H}_{R_0} \oplus L^2_{\mathrm{loc}}(\mathbf{R}^n \setminus B(0,R_0)). \end{aligned}$$

Notice that if $\chi \in C(\mathbf{R}^n)$ is constant near $\overline{B(0,R_0)}$, then $u \mapsto \chi u$ is well defined: $\mathcal{H}_{\mathrm{comp/loc}} \to \mathcal{H}_{\mathrm{comp/loc}}$.

We now consider a selfadjoint unbounded operator $P : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D} \subset \mathcal{H}$. We assume

$$u \in \mathcal{D} \Rightarrow u_{|\mathbf{R}^n\setminus B(0,R_0)} \in H^2(\mathbf{R}^n \setminus B(0,R_0)), \tag{2.32}$$

and that the restriction operator is continuous,

$$u \in H^2(\mathbf{R}^n \setminus B(0,R_0)),\ u = 0 \text{ near } \partial B(0,R_0) \Rightarrow u \in \mathcal{D}, \tag{2.33}$$

where as often we identify $H^0(\mathbf{R}^n \setminus B(0,R_0))$ with a subspace of $\mathcal{H}$ in the natural way,

$$(Pu)_{|\mathbf{R}^n\setminus B(0,R_0)} = -\Delta(u_{|\mathbf{R}^n\setminus B(0,R_0)}),\ \forall u \in \mathcal{D}. \tag{2.34}$$

Notice that if u is as in (2.33) (with vanishing $\mathcal{H}_{R_0}$ component) then $Pu = -\Delta u$ (i.e. the $\mathcal{H}_{R_0}$ component of Pu vanishes) since for every $v \in \mathcal{D}$:

$$(Pu|v) = (u|Pv) = (u| - \Delta(v_{|\mathbf{R}^n\setminus B(0,R_0)})) = (-\Delta u|v)_{L^2(\mathbf{R}^n\setminus B)}.$$

We also assume that

$$1_{B(0,R_0)}(i-P)^{-1} \text{ is compact:} \mathcal{H} \to \mathcal{H}. \tag{2.35}$$

Define $\mathcal{D}_{\mathrm{comp}} = \mathcal{D} \cap \mathcal{H}_{\mathrm{comp}}$, $\mathcal{D}_{\mathrm{loc}} = \{u \in \mathcal{H}_{\mathrm{loc}};\, \chi u \in \mathcal{D},\, \forall \chi \in C_0^\infty(\mathbf{R}^n) \text{ with } \chi = \text{Const. near } \overline{B(0,R_0)}\}$.

We should keep i mind the following two examples:

1) Let $P=-\Delta+V(x)$, where $V\in L^\infty_{\rm comp}(\mathbf{R}^n)$. Then we can take $\mathcal{H}=L^2(\mathbf{R}^n)$ and the domain of our self-adjoint operator P is $\mathcal{D}=H^2(\mathbf{R}^n)$.

2) Let $\mathcal{O}\subset\mathbf{R}^n$ be a bounded open set with smooth boundary. Let P be the unbounded self-adjoint realization of $-\Delta$ on $L^2(\mathbf{R}^n\setminus\mathcal{O})$ with domain $\mathcal{D}=\{u\in H^2(\mathbf{R}^n\setminus\mathcal{O});\, u_{|\partial\mathcal{O}}=0\}$. P is often called the exterior Dirichlet Laplacian. Similarly we can define the exterior Neumann Laplacian by taking $\mathcal{D}=\{u\in H^2(\mathbf{R}^n\setminus\mathcal{O});\, \partial_\nu u_{|\partial\mathcal{O}}=0\}$, where $\partial_\nu u$ denotes the exterior derivative in the unit normal direction.

Theorem 2.2 *a) P has only discrete spectrum in $]-\infty,0[$,*
b) $(P-\lambda^2)^{-1}:\mathcal{H}_{\rm comp}\to\mathcal{D}_{\rm loc}$ has a meromorphic extension from $\{\lambda\in\mathbf{C};\,\mathrm{Im}\,\lambda>0,\,\lambda^2\notin\sigma(P)\cap]-\infty,0[\}$ to:
$\mathbf{C}\setminus\{0\}$ when $n=1$,
$\mathbf{C}$ when $n\ge 3$ is odd,
$(\mathbf{C}\setminus\{0\})^*$ when n is even.

Here meromorphic functions with values in the space of continuous operators $\mathcal{H}_{\rm comp}\to\mathcal{D}_{\rm loc}$ are defined in the same way as in the special case of such functions with values in the continuous operators: $H^0_{\rm comp}\to H^2_{\rm loc}$.

Proof. We first "enlarge" the compactness property (2.35). If $z,w\in\rho(P)$, the resolvent set of P, then we have the resolvent identity

$$(z-P)^{-1}=(w-P)^{-1}-(w-P)^{-1}(z-w)(z-P)^{-1},\tag{2.36}$$

implying that

$$1_{B(0,R_0)}(z-P)^{-1}=1_{B(0,R_0)}(i-P)^{-1}-1_{B(0,R_0)}(i-P)^{-1}(i-w)(z-P)^{-1}\tag{2.37}$$

is compact. Passing to the adjoints, we see that $(z-P)^{-1}1_{B(0,R_0)}$ is compact too.

Let $B(x,r_0,r_1)$ denote the shell $\{y\in\mathbf{R}^n;\, r_0<|y-x|<r_1\}$. Then $1_{B(0,R_0,R)}:H^2(\mathbf{R}^n\setminus B(0,R_0))\to L^2(\mathbf{R}^n\setminus B(0,R_0))$ is compact for $R_0<R<\infty$. It follows that

$$1_{B(0,R)}(z-P)^{-1},\ (z-P)^{-1}1_{B(0,R)}:\mathcal{H}\to\mathcal{H}\tag{2.38}$$
are compact for $z\in\rho(P)$, $R_0<R<\infty$.

Let $\chi_0,\chi_1,\chi_2\in C_0^\infty$ with $1_{B(0,R_0)}\prec\chi_0\prec\chi_1\prec\chi_2$. We first let $\lambda\in\mathbf{C}$ with $\mathrm{Im}\,\lambda>0$ and take μ with the same properties and such that $\mu^2\notin\sigma(P)$. We will fix μ later. Put

$$Q(\lambda,\mu)=(1-\chi_0)R_0(\lambda)(1-\chi_1)+\chi_2R(\mu)\chi_1=Q_0(\lambda,\mu)+Q_1(\lambda,\mu):\ \mathcal{H}\to\mathcal{D},$$

where $R_0(\lambda) = (P_0 - \lambda^2)^{-1}$, $R(\mu) = (P - \mu^2)^{-1}$, and the last equality above gives the obvious definition of Q_0, Q_1. Then:

$$(P - \lambda^2)Q_0 = (1 - \chi_1) + [\Delta, \chi_0]R_0(\lambda)(1 - \chi_1) = (1 - \chi_1) + K_0(\lambda), \quad (2.39)$$

$$\begin{aligned}(P - \lambda^2)Q_1 &= \chi_2(P - \lambda^2)R(\mu)\chi_1 - [\Delta, \chi_0]R_0(\mu)\chi_1 \qquad (2.40)\\ &= \chi_1 + \chi_2(\mu^2 - \lambda^2)R(\mu)\chi_1 - [\Delta, \chi_2]R(\mu)\chi_1 = \chi_1 + K_1(\lambda, \mu),\end{aligned}$$

where the last equations define the operators K_0, K_1. Hence,

$$(P - \lambda^2)Q = 1 + K_0(\lambda) + K_1(\lambda, \mu) = 1 + K(\lambda, \mu), \quad (2.41)$$

where the last equality defines K.

Using (2.38) and the fact that $[\Delta, \chi_j]$ is a first order operator with smooth compactly supported coefficients and therefore compact: $H^2(\mathbf{R}^n \backslash B(0, R_0)) \to L^2(\mathbf{R}^n \setminus B(0, R_0))$, we see that

$$K(\lambda, \mu) \text{ is compact.} \quad (2.42)$$

Now choose μ with $\arg \mu = \frac{\pi}{4}$, so that $\operatorname{Im}(\mu^2) = |\mu|^2$. Then

$$R(\mu) = \begin{cases} \mathcal{O}(\frac{1}{|\mu|^2}) : \mathcal{H} \to \mathcal{H}, \\ \mathcal{O}(1) : \mathcal{H} \to \mathcal{D}, \end{cases}$$

and,

$$(1 - \chi_0)R(\mu) = \begin{cases} \mathcal{O}(\frac{1}{|\mu|^2}) : \mathcal{H} \to L^2 \\ \mathcal{O}(1) : \mathcal{H} \to H^2, \end{cases}$$

so by interpolation, $(1 - \chi_0)R(\mu) = \mathcal{O}(\frac{1}{|\mu|}) : \mathcal{H} \to H^1$. It follows that

$$K(\mu, \mu) = \mathcal{O}(\frac{1}{|\mu|}) : \mathcal{H} \to \mathcal{H}, \quad (2.43)$$

and consequently $1 + K(\mu, \mu) : \mathcal{H} \to \mathcal{H}$ is invertible with a bounded inverse $(1 + K(\mu, \mu))^{-1}$ of norm ≤ 2 when $|\mu|$ is large enough (and $\arg \mu = \pi/4$). We fix such a value of μ and apply analytic Fredholm theory with respect to λ. It follows that $(1 + K(\lambda, \mu))^{-1}$ exists except for λ in a discrete set in the open upper half plane, and that the poles are of finite order and with coefficients of finite rank for the singular terms in the Laurent expansion at each such pole. (See chapter 5.)

Writing

$$(P-\lambda^2)^{-1} = Q(\lambda,\mu)(1+K(\lambda,\mu))^{-1}, \tag{2.44}$$

we see that $(P-\lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$ is meromorphic in the upper half plane with only finite rank singular terms in the Laurent expansion. Of course, we already know that the only possible singularities are situated on the negative imaginary axis and correspond to the negative spectrum of P, so we see that that this spectrum is purely discrete and we have obtained part (a) of the theorem. (The spectral projection onto any finite part of the spectrum is given by $\frac{1}{2\pi i}\int_\gamma (\lambda^2 - P)^{-1} d(\lambda^2)$, where γ is simple closed contour in the upper half plane which avoids the poles and encircles a certain number of square roots of negative eigenvalues.)

We next consider the meromophic extension and let $n \geq 3$ be odd in order to fix the ideas. We keep μ fixed as before. Then $\mathbf{C} \ni \lambda \mapsto Q(\lambda,\mu) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$ is holomorphic. Let $\chi_2 \prec \chi \in C_0^\infty(\mathbf{R}^n)$, so that $\chi K = K$. To invert $1+K : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{comp}}$ it suffices to invert $1 + K\chi : \mathcal{H} \to \mathcal{H}$. Indeed, we observe that $(1-\chi)K = 0$ and hence

$$(1+K(1-\chi))^{-1} = 1 - K(1-\chi),\ (1+K) = (1+K(1-\chi))(1+K\chi),$$

implying

$$(1+K)^{-1} = (1+K\chi)^{-1}(1-K(1-\chi)). \tag{2.45}$$

Using (2.45) in (2.44) and applying analytic Fredholm theory to $(1+K\chi)^{-1}$, we get the desired meromorphic extension

$$R(\lambda) = Q(\lambda,\mu)(1+K)^{-1} = Q(1+K\chi)^{-1}(1-K(1-\chi)). \tag{2.46}$$

#

In the following we shall sometimes use the notation $R(\lambda) = (P-\lambda)^{-1}$ for the meromorphic extension of the resolvent, even though it is no more an inverse, strictly speaking. The poles of the meromorphic extension not already in the physical halfplane $\mathbf{C}_+ := \{\lambda \in \mathbf{C}; \operatorname{Im}\lambda > 0\}$ that was the starting point of the extension, will be called resonances (and sometimes scattering poles). If λ_0 is a non-vanishing resonance, we define its multiplicity $m(\lambda_0)$ by

$$m(\lambda_0) = \operatorname{rank} \frac{1}{2\pi i}\int_\gamma (\lambda^2 - P)^{-1} d(\lambda^2), \tag{2.47}$$

where γ is a small positively oriented circle centered at λ_0. With $z_0 = \lambda_0^2$, $z = \lambda^2$, we have for z close to z_0:

$$(z - P)^{-1} = (z - z_0)^{-N}\Pi_{-N} + .. + (z - z_0)^{-1}\Pi_{-1} + \mathrm{Hol}\,(z) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}\,, \tag{2.48}$$

where the last term is holomorphic in a neighborhood of $z = z_0$. From this expression we see that $m(\lambda_0)$ is the rank of Π_{-1} in view of the standard formula:

$$\Pi_{-j} = \frac{1}{2\pi i}\int_{\widetilde{\gamma}}(z - z_0)^{j-1}(z - P)^{-1}dz,\ j \geq 1. \tag{2.49}$$

Here $\widetilde{\gamma}$ denotes the contour in the z plane which corresponds to γ in the λ plane.

Use that

$$(P - z_0)(z - P)^{-1} = (z - z_0)(z - P)^{-1} - 1$$

by holomorphic extension from the physical sheet, to get for $j \geq 1$:

$$(P - z_0)\Pi_{-j} = \frac{1}{2\pi i}\int_{\widetilde{\gamma}}(z - z_0)^{j-1}(P - z_0)(z - P)^{-1}dz \tag{2.50}$$

$$= \frac{1}{2\pi i}\int_{\widetilde{\gamma}}(z - z_0)^{j}(z - P)^{-1}dz - \frac{1}{2\pi i}\int_{\widetilde{\gamma}}(z - z_0)^{j-1}dz = \Pi_{-(j+1)}$$

Consequently $(P - z_0)$ maps $\Pi_{-j}(\mathcal{H}_{\mathrm{comp}})$ into $\Pi_{-(j+1)}(\mathcal{H}_{\mathrm{comp}})$ (also included in $\mathcal{D}_{\mathrm{loc}}$). Since Π_{-j} are of finite rank and $\mathcal{D}_{\mathrm{comp}}$ is dense in $\mathcal{H}_{\mathrm{comp}}$ [[détailler cet argument]], we have $\Pi_{-j}(\mathcal{H}_{\mathrm{comp}}) = \Pi_{-j}(\mathcal{D}_{\mathrm{comp}})$. Moreover, on $\mathcal{D}_{\mathrm{comp}}$ we have $\Pi_{-j}P = P\Pi_{-j}$, so (2.50) becomes:

$$\Pi_{-(j+1)} = \Pi_{-j}(P - z_0) \text{ on } \mathcal{D}_{\mathrm{comp}}. \tag{2.51}$$

Consequently,

$$\Pi_{-(j+1)}(\mathcal{H}_{\mathrm{comp}}) = \Pi_{-(j+1)}(\mathcal{D}_{\mathrm{comp}}) \subset \Pi_{-j}(\mathcal{H}_{\mathrm{comp}}),$$

and $(P - z_0)_{|\Pi_{-1}(\mathcal{H}_{\mathrm{comp}})}$ is a nilpotent operator. The elements of the kernel of this operator will be called resonant states. The elements of $\Pi_{-1}(\mathcal{H}_{\mathrm{comp}})$ will be called generalized resonant states.

If $\Pi_{-1}v$ is a resonant state, then $0 = \Pi_{-2}v = \Pi_{-3}v = ..$ and $(z - P)^{-1}v$ has a simple pole at z_0.

Proposition 2.3 *Let* $1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbf{R}^n)$. *Then with* γ *as in (2.47),*

$$m(\lambda_0) = \mathrm{rank} \int_\gamma (\lambda^2 - P)^{-1} \chi d(\lambda^2) \tag{2.52}$$

Proof. Clearly $m(\lambda_0)$ is at least as large as the RHS of (2.52). Let $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^n)$, with $1_{B(0,R_0)} \prec \widetilde{\chi} \prec \chi$. From (2.44), we get

$$(P - \lambda^2)^{-1} = Q - (P - \lambda^2)^{-1} K(\lambda, \mu) = Q - (P - \lambda^2)^{-1} \widetilde{\chi} K(\lambda, \mu),$$

provided we let the cutoffs χ_1, χ_2, χ_3 have their supports sufficiently close to $B(0, R_0)$. Hence

$$\frac{1}{2\pi i} \int_\gamma (\lambda^2 - P)^{-1} d(\lambda^2) = -\frac{1}{2\pi i} \int_\gamma (\lambda^2 - P)^{-1} \widetilde{\chi} K(\lambda, \mu) d(\lambda^2). \tag{2.53}$$

Recall that μ is fixed and write the Taylor expansion of K at λ_0^2 in the variable λ^2:

$$K(\lambda, \mu) = \sum_{j=0}^{N-1} K_j(\mu)(\lambda^2 - \lambda_0^2)^j + (\lambda^2 - \lambda_0^2)^N)\mathrm{Hol}\,(\lambda),$$

where $\mathrm{Hol}\,(\lambda)$ is function which is holomorphic for λ close to λ_0. Use this in (2.53) together with (2.49):

$$\frac{1}{2\pi i} \int_\gamma (\lambda^2 - P)^{-1} d(\lambda^2) = \tag{2.54}$$
$$- \sum_0^{N-1} \frac{1}{2\pi i} \int_\gamma (\lambda^2 - \lambda_0^2)^j (\lambda^2 - P)^{-1} d(\lambda^2) \widetilde{\chi} K_j = -\sum_0^{N-1} \Pi_{-(1+j)} \widetilde{\chi} K_j.$$

Here we use (2.50):

$$\Pi_{-(j+1)} \widetilde{\chi} = \Pi_{-j}(P - z_0)\widetilde{\chi} = \Pi_{-j}\widehat{\chi}(P - z_0)\widetilde{\chi},$$

if $\widetilde{\chi} \prec \widehat{\chi}$, so

$$\mathcal{R}(\Pi_{-(j+1)}\widetilde{\chi}) = \mathcal{R}(\Pi_{-j}\widehat{\chi}).$$

Iterating, we get

$$\mathcal{R}(\Pi_{-(j+1)}\widetilde{\chi}) \subset \mathcal{R}(\Pi_{-1}\chi),$$

and using this in (2.54), we get

$$\mathcal{R}(\Pi_{-1}) \subset \mathcal{R}(\Pi_{-1}\chi),$$

which implies (2.52). #

2.4 Absence of real resonances.

We end this chapter by discussing the relation with outgoing solutions to the free Helmholtz equation, and then applying it to show that real non-vanishing resonances will not appear in standard situations. (Rellich's theorem.) We shall first see that resonant states associeted to such resonances are outgoing states in a sense that we define below.

Let $1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbf{R}^n)$. For $z = \lambda^2$, $\operatorname{Im} \lambda > 0$, we have for $v \in \mathcal{H}_{\mathrm{comp}}$ with $1_{\operatorname{supp} v} \prec \chi$:

$$(z - P)^{-1} v = \chi (z - P)^{-1} v + (z - P_0)^{-1} [P, \chi] (z - P)^{-1} v, \tag{2.55}$$

where we notice that the second term of the RHS vanishes outside $\operatorname{supp}(1 - \chi)$. In fact, let $u = (z - P)^{-1} v$ and compute

$$(z - P_0)((1 - \chi)u) = (z - P)((1 - \chi)u) = [P, \chi]u$$

which implies that

$$(1 - \chi)(z - P)^{-1} v = (z - P_0)^{-1} [P, \chi](z - P)^{-1} v.$$

The relation (2.55) extends by analytic continuation to the non-physical sheet(s). Let $z_0 = \lambda_0^2$ where λ_0 is a resonance. Integrating (2.55) along a small contour γ around z_0, we get for $z \in \mathbf{R}^n \setminus \overline{B(0, R_0)}$:

$$\begin{aligned} \Pi_{-1} v - \chi \Pi_{-1} v &= \sum_{j=1}^{N} \frac{1}{2\pi i} \int (z - P_0)^{-1} [P, \chi] (z - z_0)^{-j} \Pi_{-j} v dz \\ &= \sum_{j=1}^{N} \frac{1}{(j-1)!} (\partial_z^{j-1} (z - P_0)^{-1})_{z = z_0} [P, \chi] \Pi_{-j} v. \end{aligned} \tag{2.56}$$

Choose $v \in \mathcal{H}_{\mathrm{comp}}$ so that $\Pi_{-1} v$ is a resonant state. Then (2.56) simplifies to

$$\Pi_{-1} v = \chi \Pi_{-1} v + (z_0 - P_0)^{-1} [P, \chi] \Pi_{-1} v, \tag{2.57}$$

where we recall that $(z_0 - P_0)^{-1}$ denotes the holomorphic extension from the physical sheet to the point z_0.

For simplicity, we now restrict the attention to a real positive $z_0 = z$ or equivalently to a real non-vanishing λ. Let $R_0(\lambda) = (P_0 - \lambda^2)^{-1}$ be the branch of the resolvent obtained by extension from the upper half plane: $R_0(\lambda) = (P_0 - (\lambda + i0)^2)^{-1} : \mathcal{E}' \to \mathcal{S}'$.

Definition. We say that $u \in \mathcal{S}'(\mathbf{R}^n)$ is outgoing if there exists $w \in \mathcal{E}'(\mathbf{R}^n)$ such that

$$u(x) = R_0(\lambda)w \text{ for } |x| \gg 1. \tag{2.58}$$

We have just seen that a resonant state (or rather it's component in $L^2_{\mathrm{loc}}(\mathbf{R}^n \setminus B(0, R_0)))$ is outgoing.

Exercise: Let $u \in \mathcal{S}(\mathbf{R}^n)$ be outgoing, so that $u(x) = R_0(\lambda)w(x)$, $|x| \gg 1$, for some $w \in \mathcal{E}'$. Then $(P_0 - \lambda^2)u = v \in \mathcal{E}'$. Show that $u = R_0(\lambda)v$.

Solution: As in (2.55) we have for $\chi \in C_0^\infty$ with $\chi = 1$ on a sufficiently large ball:

$$R_0(\lambda)w = \chi R_0(\lambda)w - R_0(\lambda)[P_0, \chi]R_0(\lambda)w,$$

and hence $u = -R_0(\lambda)[P_0, \chi]u$ near ∞.

On the other hand, let $1_{\mathrm{supp}\, v} \prec \chi$. Then

$$(P_0 - \lambda^2)\chi u = v + [P_0, \chi]u \in \mathcal{E}', \tag{2.59}$$

and on $\mathcal{E}'$, we have $R_0(\lambda)(P_0 - \lambda^2) = 1$ by analytic extension from the physical half plane. So applying $R_0(\lambda)$ to (2.59), we get

$$\chi u = R_0(\lambda)v + R_0(\lambda)[P_0, \chi]u.$$

Hence

$$R_0(\lambda)v = -R_0(\lambda)[P_0, \chi]u = u \text{ near } \infty.$$

$u - R_0(\lambda)v$ is therefore in $\mathcal{E}'$ and since it satisfies $(P_0 - \lambda^2)(u - R_0(\lambda)v) = 0$, it has to be 0. (Notice or recall that every non-vanishing partial differential operator with constant coefficients is injective $\mathcal{E}'(\mathbf{R}^n) \to \mathcal{E}'(\mathbf{R}^n)$.) #

Theorem 2.4 *If $\lambda \in \mathbf{R} \setminus \{0\}$ is a resonance and $u \in \mathcal{D}_{\mathrm{loc}}$ a corresponding resonant state, then u has compact support.*

We notice that in most standard situations, there cannot be any non-trivial solution of the equation $(P - \lambda^2)u$ which has compact support and hence the theorem implies that that there are no real non-vanishing resonances. For the proof of the theorem we need

Proposition 2.5 *Let* $v \in C_0^\infty(\mathbf{R}^n)$*,* $1_{\mathrm{supp}\, v} \prec \chi \in C_0^\infty(\mathbf{R}^n;\mathbf{R})$*,* $0 \neq \lambda \in \mathbf{R}$*. Then*

$$\frac{1}{2i}([-\Delta,\chi]R_0(\lambda)v|R_0(\lambda)v) = C(\lambda)\int_{\partial B(0,|\lambda|)} |\widehat{v}(\xi)|^2 L(d\xi) = \mathrm{Im}\,(R_0(\lambda)v|v), \tag{2.60}$$

where $C(\lambda)\neq 0$ *can be computed, and* $L(d\xi)$ *denotes the standard Lebesgue measure on the sphere of radius* $|\lambda|$*.*

Proof. This is a simple computation, where we assume that $\lambda > 0$ for simplicity:

$$\begin{aligned}
\frac{1}{2i}([-\Delta,\chi]R_0(\lambda)v|R_0(\lambda)v) &= \frac{1}{2i}([-\Delta-\lambda^2,\chi]R_0(\lambda)v|R_0(\lambda)v)\\
= \frac{1}{2i}\Big((\chi R_0(\lambda)v|(-\Delta-\lambda^2)R_0(\lambda)v) &- (\chi(-\Delta-\lambda^2)R_0(\lambda)v|R_0(\lambda)v)\Big)\\
&= \frac{1}{2i}\Big((R_0(\lambda)v|\chi v) - (\chi v|R_0(\lambda)v)\Big)\\
&= \mathrm{Im}\,(R_0(\lambda)v|\chi v) = \mathrm{Im}\,(R_0(\lambda)v|v) =\\
\mathrm{Im}\,\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{\widehat{v}(\xi)\overline{\widehat{v}}(\xi)}{(\xi^2-(\lambda+i0)^2)}d\xi &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}|\widehat{v}(\xi)|^2\,\mathrm{Im}\,\frac{1}{(|\xi|^2-(\lambda+i0)^2)}d\xi\\
&= \frac{\pi}{(2\pi)^n 2\lambda}\int_{\partial B(0,\lambda)}|\widehat{v}(\xi)|^2 L(d\xi).
\end{aligned}$$

#

Notice that by density the first equality in (2.60) holds also for $v \in \mathcal{E}'$.

Proof of Theorem 2.4. Let u be an outgoing solution of $(P-\lambda^2)u = 0$ with $\lambda \in \mathbf{R}_+$. Let $1_{\overline{B(0,R_0)}} \prec \chi \in C_0^\infty$ be real-valued and consider

$$\begin{aligned}
\frac{1}{2i}([P,\chi]u|u) &= \frac{1}{2i}((\chi u|Pu) - (\chi Pu|u))\\
&= \frac{1}{2i}(\lambda^2(\chi u|u) - \lambda^2(\chi u|u)) = 0.
\end{aligned}$$

On the other hand, if we write $u = R_0(\lambda)v$, $v \in C_0^\infty$, for $|x| \gg 1$, (we can take $v = -[P_0,\widetilde{\chi}]u$ for a suitable $\widetilde{\chi}$), then if we let $\chi \in C_0^\infty$ be a real-valued cut-off which is equal to 1 on a sufficiently large ball, we have

$$\begin{aligned}
0 = (\frac{1}{2i}[P,\chi]u|u) &= \frac{1}{2i}([-\Delta,\chi]R_{0,+}(\lambda)v|R_{0,+}(\lambda)v)\\
&= C(\lambda)\int_{\partial B(0,\lambda)}|\widehat{v}(\xi)|^2 L(d\xi),
\end{aligned}$$

so $\widehat{v}(\xi) = 0$ on $B(0, \lambda)$ and hence also on the complexification of this ball. By the Paley-Wiener theorem plus easy division estimates (see Lemma 2.6 below), we see that $R_{0,+}(\lambda)v \in \mathcal{E}'$. Hence u has compact support. #

Remark. In many situations, we know that there are no non-trivial solutions of $(P - \lambda^2)u = 0$ with compact support, and it then follows from Theorem 2.4 that there are no real non-vanishing resonances.

Lemma 2.6 *Let $v \in \mathcal{E}'(\mathbf{R}^n)$ with $\widehat{v}(\xi) = 0$ on the real sphere $\xi^2 = \lambda^2$, for some $\lambda > 0$. Then $R_0(\lambda)v \in \mathcal{E}'(\mathbf{R}^n)$ (which implies that $\operatorname{supp} R_0(\lambda)v$ is contained in the union of the support of v and the bounded components of the complement of the support of v).*

Proof. We recall a weak form of the Paley-Wiener Theorem: If $u \in \mathcal{S}'(\mathbf{R}^n)$, then u belongs to $\mathcal{E}'$ if and only if

$$\begin{aligned} &\exists C_0,\, C_1,\, N_0 > 0, \text{ such that } \widehat{u}(\xi) \text{ is entire and} \\ &|\widehat{u}(\xi)| \le C_1 \langle \xi \rangle^{N_0} e^{C_0 |\operatorname{Im} \xi|},\ \xi \in \mathbf{C}^n. \end{aligned} \tag{2.61}$$

If v satisfies the assumptions of the lemma, then $\widehat{u} := \frac{\widehat{v}(\xi)}{(\xi^2 - \lambda^2)}$ is an entire function and it is easy to see that $\widehat{u}$ is the Fourier transform of $R_0(\lambda)v$. It suffices, then to show that $\widehat{u}(\xi)$ satisfies (2.61) for some suitable constants.

Let $\delta > 0$ be small (to be fixed later) and let $\xi_0 \in \mathbf{C}^n$ with $|\xi_0^2 - \lambda^2| \le \delta$. Write

$$\xi^2 - \lambda^2 = 2\xi_0(\xi - \xi_0) + (\xi_0^2 - \lambda^2) + (\xi - \xi_0)^2.$$

Take

$$\xi - \xi_0 = z\frac{\overline{\xi}_0}{|\xi_0|}, \text{ with } z \in \mathbf{C},\ |z| = \epsilon > 0,$$

where $\epsilon > 0$ will be fixed independently of ξ_0.

Then

$$|\xi^2 - \lambda^2| \ge 2|\xi_0|\epsilon - \delta - \epsilon^2.$$

λ being fixed, we first choose $\delta \le \delta_1(\lambda)$ small enough, so that $|\xi_0| \ge \frac{|\lambda|}{2}$ and get

$$|\xi^2 - \lambda^2| \ge |\lambda|\epsilon - \delta - \epsilon^2.$$

Choose $\epsilon < |\lambda|/2$ so that $|\xi^2 - \lambda^2| \ge |\lambda|\epsilon/2 - \delta$ and finally $\delta \le \delta_2(\lambda, \epsilon)$ so that $|\xi^2 - \lambda^2| \ge |\lambda|\epsilon/4$. Consider

$$\widehat{u}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|}) = \frac{\widehat{v}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})}{(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})^2 - \lambda^2}$$

which is holomorphic for $|z| < 2\epsilon$. We have

$$|\widehat{u}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})| \leq 4\frac{|\widehat{v}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})|}{|\lambda|\epsilon}$$

for $|z| = \epsilon$, so by the maximum principle we get for $|z| \leq \epsilon$:

$$|\widehat{u}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})| \leq \frac{4}{|\lambda|\epsilon}\sup_{|z|=\epsilon}|\widehat{v}(\xi_0 + z\frac{\overline{\xi}_0}{|\xi_0|})|.$$

Especially we get the last estimate for $\widehat{u}(\xi_0)$, and we conclude that if $\widehat{v}$ satisfies (2.61), then $\widehat{u}$ also satisfies (2.61) in the region $|\xi^2 - \lambda^2| \leq \delta$, and hence everywhere, since $|\widehat{u}(\xi)| \leq \delta^{-1}|\widehat{v}(\xi)|$ in the region where $|\xi^2 - \lambda^2| \geq \delta$. #

Remark. It is a little easier to show that if

$$e_0 \in \mathcal{D},\ (P - \lambda^2)e_0 = 0,\ \lambda \in \mathbf{R} \setminus \{0\},$$

so that λ^2 is a positive eigenvalue, then e_0 has compact support. Take χ as usual, equal to one near $\overline{B(0, R_0)}$. Then

$$(P_0 - \lambda^2)(1 - \chi)e_0 = [\Delta, \chi]e_0 \in H^1_{\mathrm{comp}},$$

so if $\mathcal{F}$ denotes the Fourier transformation, then

$$(\xi^2 - \lambda^2)\mathcal{F}((1 - \chi)e_0)(\xi) = (\mathcal{F}[\Delta, \chi]e_0)(\xi).$$

Here the right hand side is entire and in order to have $\mathcal{F}(1 - \chi)e_0 \in L^2$, we must have $\mathcal{F}[\Delta, \chi]e_0)(\xi) = 0$ when $\xi^2 = \lambda^2$. As before, we conclude by means of Paley-Wiener, that

$$(\mathcal{F}((1 - \chi)e_0)(\xi) = \frac{\mathcal{F}([\Delta, \chi]e_0)(\xi))}{\xi^2 - \lambda^2}$$

is an entire function and satisfies (2.61). Hence $(1-\chi)e_0$ has compact support and so does e_0.

[[[Missing piece in this presentation: Show that if $(P - \lambda^2)u = 0$ and u is outgoing, then u is a resonant state and λ is a resonance.]]]

3 Expansion of solutions to the wave equation in exponentially decaying resonant modes, when $t \to +\infty$.

In this chapter we show (in the case of odd dimension $n \geq 3$) that if a certain non-trapping assumption for the associated wave equation is fulfilled, then for every $N > 0$ there are at most finitely many resonances above the curve

$$\operatorname{Im} z = -N \log_+ |\operatorname{Re} z|$$

(This fact is true also in even dimensions, but a little more difficult to prove in that case.) Moreover if the space dimension is *odd* and ≥ 3 we shall show that the solution of the associated wave equation with compactly supported initial data has an asymptotic expansion in exponentially decaying modes: $e^{-i\lambda_j t} u_j(x,t)$ as $t \to +\infty$, where λ_j is a resonance or a square root of a negative eigen-value and u_j is a polynomial of degree $\leq \omega(\lambda_j) - 1$ in t, where $\omega(\lambda_j)$ is the order of λ_j as a pole of the meromorohic extension of $(P - \lambda^2)^{-1}$. This result is due to Lax-Phillips (see [50]) in a more restricted setting, and the idea of the proof we give is due to Vainberg [91]. Vainberg's proof was recently adapted to the black box setting by Tang and Zworski [90], but our proof (and result) differs slightly. At one point we shall use a basic result on the propagation of singularities for the wave operator. See [42].

Let P be a bb operator as in the preceding chapter, satisfying the additional assumption:

$$P \geq -C, \tag{3.1}$$

for some constant C. Let $\mathcal{D}^j$ denote the domain of $\langle P \rangle^j$ when $j \geq 0$ and let $\mathcal{D}^j = (\mathcal{D}^{-j})^*$ be the dual space of $\mathcal{D}^{-j}$, when $j \leq 0$.

Recall that by the spectral theorem, P is unitarily equivalent to the multiplication operator $\widetilde{P}: u(m) \mapsto p(m)u(m)$ in $\widetilde{H} = L^2(M, \mu)$ where M is some set, μ a measure on M and $p(m)$ a real-valued measurable function on M. Then $\mathcal{D}^j$ becomes $\{\langle p(m) \rangle^{-j} u(m);\, u \in L^2(M; \mu)\}$. From this representation and the Hölder inequality, we deduce the interpolation inequality

$$\|u\|_{\mathcal{D}^{\theta j + (1-\theta)k}} \leq \|u\|_{\mathcal{D}^j}^{\theta} \|u\|_{\mathcal{D}^k}^{1-\theta},\ j,k \in \mathbf{R},\, 0 \leq \theta \leq 1,\, u \in \mathcal{D}^{\max(j,k)}. \tag{3.2}$$

For $g \in \mathcal{D}^j$, we can solve the abstract Cauchy problem

$$(P + \partial_t^2) u = 0,\ u(0) = 0,\, \partial_t u(0) = g, \tag{3.3}$$

by the functional formula:

$$u(t) = \frac{\sin t\sqrt{P}}{\sqrt{P}} g =: U(t)g. \tag{3.4}$$

(If we apply the unitary equivalence above, then $U(t)$ can be identified with the multiplication operator $u(m) \mapsto \frac{\sin(t\sqrt{p(m)})}{\sqrt{p(m)}} u(m)$.) Note that $f(t,E) := \frac{\sin t\sqrt{E}}{\sqrt{E}}$ is a well-defined smooth function on $\mathbf{R}\times\mathbf{R}$, independent of the choice of branch of $\sqrt{E}$, which satisfies the following estimates for $E \geq -\text{Const.}$:

$$\partial_t^k f(t,E) = \mathcal{O}_k(1) e^{|t|C} \langle E\rangle^{(k-1)/2}. \tag{3.5}$$

It follows that

$$U(t) \in C^m(\mathbf{R}_t; \mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j+\frac{1-m}{2}})), \tag{3.6}$$

where the mth derivative is merely strongly continuous in t with values in $\mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j-\frac{m-1}{2}})$, while the derivatives of lower order are continuous in the norm of $\mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j+\frac{1-m}{2}})$. Moreover,

$$\|\partial_t^k U\|_{\mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j+\frac{1-k}{2}})} \leq C_{k,j} e^{C|t|}. \tag{3.7}$$

We take a break in the general discussion in order to recall some basic existence and uniqueness results for the abstract equation (3.3).

Proposition 3.1 *If $v \in C^0(\mathbf{R}; \mathcal{D}^{j-\frac{1}{2}})$, $f \in \mathcal{D}^j$, $g \in \mathcal{D}^{j-\frac{1}{2}}$, then*

$$u(t) := U'(t)f + U(t)g + \int_0^t U(t-s)v(s)ds$$

belongs to $C^0(\mathbf{R}; \mathcal{D}^j) \cap C^1(\mathbf{R}; \mathcal{D}^{j-\frac{1}{2}}) \cap C^2(\mathbf{R}; \mathcal{D}^{j-1})$ and solves the Cauchy problem,

$$(\partial_t^2 + P)u = v, \ u(0) = f, \ u'(0) = g. \tag{3.8}$$

Proof. Notice that $U(0) = 0$, $U'(0) = 1$, so $u_1 := U'(t)f + U(t)g$ solves (3.8) with $v = 0$. Let

$$u_2(t) = \int_0^t U(t-s)v(s).$$

Then $u_2(0) = 0$,

$$u_2'(t) = U(0)v(t) + \int_0^t U'(t-s)v(s)ds = \int_0^t U'(t-s)v(s)ds,$$

so $u_2'(0) = 0$. Since $U'(0) = 1$, we get

$$\partial_t^2 u_2(t) = v(t) + \int_0^t U''(t-s)v(s)ds,$$

and hence

$$\partial_t^2 u_2(t) + Pu_2(t) = v(t) + \int_0^t (\partial_t^2 + P)(U(t-s)v(s))ds = v(t),$$

since $(\partial_t^2 + P)(U(t-s)v(s)) = 0$. Further it is easy to see that $u_2(t)$ belongs to the required space. #

Next look at the uniqueness of solutions of (3.8):

Proposition 3.2 *Let I be an interval containing 0, and let $u \in C^2(I; \mathcal{D}^k)$ be a solution of $(\partial_t^2 + P)u = 0$. Then*

$$u(t) = U'(t)u(0) + U(t)u'(0). \tag{3.9}$$

Proof. Let $t \in I$, and put $v(s) = U(s-t)\phi$, $\phi \in \mathcal{D}^N$ for some sufficiently large N. Then for s between 0 and t:

$$\begin{aligned}\frac{d}{ds}[(u'(s)|v(s)) - (u(s)|v'(s))] &= (u''(s)|v(s)) - (u(s)|v''(s)) \\ &= -(Pu(s)|v(s)) + (u(s)|Pv(s)) = 0.\end{aligned}$$

Hence

$$(u'(t)|v(t)) - (u(t)|v'(t)) = (u'(0)|v(0)) - (u(0)|v'(0)),$$

i.e.

$$-(u(t)|\phi) = (u'(0)|U(-t)\phi) - (u(0)|U'(-t)\phi).$$

$U(t)$ is odd in t, so $U(-t) = -U(t)$, $U'(-t) = U'(t)$. Hence

$$(u(t)|\phi) = (u'(t)|U(t)\phi) + (u(0)|U'(t)\phi).$$

Use also that $U(t)$ and $U'(t)$ are symmetric:

$$(u(t)|\phi) = (U(t)u'(0) + U'(t)u(0)|\phi).$$

Since $\mathcal{D}^N$ is dense in $\mathcal{D}^{-k}$, the space dual to $\mathcal{D}^k$, we get (3.9). #

We now return to the main discussion. Outside $\overline{B(0, R_0)}$, (3.3) is just the standard wave equation, for which Huygens' principle applies and says that supports of solutions cannot propagate with speed > 1: If $u \in C^2(\mathbf{R}; \mathcal{D}'(\mathbf{R}^n))$ and

$$(-\Delta + \partial_t^2)u = 0,\ u(0) = u_0,\ \partial_t u(0) = u_1, \tag{3.10}$$

then

$$\operatorname{supp} u \subset \{(t, x);\ \operatorname{dist}_{\mathbf{R}^n}(x, \operatorname{supp} u_0 \cup \operatorname{supp} u_1) \le |t|\}. \tag{3.11}$$

In the bb setting, we define the support of $u \in \mathcal{H}_{\mathrm{loc}}$ to be $\operatorname{supp} u_{|\mathbf{R}^n \setminus B(0,R_0}$ if u vanishes near $\overline{B(0, R_0)}$ and $\overline{B(0, R_0)} \cup \operatorname{supp} u_{|\mathbf{R}^n \setminus B(0,R_0)}$ otherwise. Notice that $\operatorname{supp} u$ is closed.

Let $\operatorname{dist}_{bb}(x, y) = \min(|x-y|, \operatorname{dist}(x, B(0, R_0)) + \operatorname{dist}(y, B(0, R_0)))$. Then for (3.3) we have

$$\operatorname{supp} u \subset \{(t, x); \operatorname{dist}_{bb}(x, \operatorname{supp} g) \le |t|\}. \tag{3.12}$$

We now introduce an abstract non-trapping condition, which says that singularities in compactly supported initial data are washed away from any bounded region after a suitable time:

$$\begin{aligned} &\text{For all } a > R_0,\ \exists T_a > 0,\ \text{such that } t \mapsto \chi U(t)\chi \\ &\text{belongs to } C^\infty(]T_a, T_a + 1[; \mathcal{L}(\mathcal{D}^0; \mathcal{D}^N)) \\ &\text{for every } N \ge 0 \text{ and every } \chi \in C_0^\infty(B(0, a)) \text{ with } 1_{B(0,R_0)} \prec \chi. \end{aligned} \tag{3.13}$$

In practice, the verification of this non-trapping condition is important and depends on some result on the propagation of singularities for solutions of the wave equation. The condition (3.13) is known to hold in the following two cases: 1) $P = -\Delta + V$, $V \in C_0^\infty(\mathbf{R}^n)$ and 2) $-P$ is the exterior Dirichlet or Neumann Laplacian on $\mathbf{R}^n \setminus \mathcal{O}$ where $\mathcal{O} \subset\subset \mathbf{R}^n$ is open with smooth boundary and non-trapping in the geometric sense that no maximal optical ray can be contained in a bounded set. For the second example we refer to [56], while the first example follows from the general result of Hörmander on progation of singularities for solutions of equations of principal type. See [42].

Let $a > R_0 + 1$ and let $1_{\overline{B(0,R_0}} \prec \chi_{a-1} \in C_0^\infty(B(0, a-1))$. We will work for $t \ge 0$ in the following. Let $\psi_a(t) \in C_0^\infty(] -\infty, T_a + 1[; [0, 1])$ be equal to 1 near $] - \infty, T_a]$. Then

$$\begin{aligned} &(P + \partial_t^2)(\psi_a(t)U(t)\chi_{a-1}) = \\ &(\partial_t^2(\psi_a(t)) + 2\partial_t(\psi_a(t))\partial_t)U(t)\chi_{a-1} =: V_a(t). \end{aligned} \tag{3.14}$$

In view of (3.13), we see that if $1_{\overline{B(0,R_0)}} \prec \chi \in C_0^\infty(B(0,a))$, then

$$\chi V_a(t) \in C_0^\infty(]T_a, T_a+1[; \mathcal{L}(\mathcal{H}, \mathcal{D}^N)),\ \forall N \in \mathbf{N}. \tag{3.15}$$

We will choose $\chi \prec \chi_{a-1}$.

The "exterior part" $(1-\chi)V_a(t)g$ will not be smooth in general, and we will add a correction for it, by solving the free wave equation. Let $U_0(t)$ be the free wave "group" defined as in (2.7) (with P replaced by $-\Delta$ and with $\mathcal{H}$ replaced by $L^2(\mathbf{R}^n)$). As explained in chapter 2, we have the corresponding forward fundamental solution E_0 given by

$$E_0 = H(t)U_0(t), \tag{3.16}$$

so that $(\partial_t^2 - \Delta)E_0 = \delta_{\mathbf{R}^{n+1}}$, and

$$\operatorname{supp} E_0 = \begin{cases} \{(t,x) \in \mathbf{R}\times\mathbf{R}^n;\ |x| = t\}, & \text{for } n \text{ odd } \geq 3, \\ \{(t,x) \in \mathbf{R}\times\mathbf{R}^n;\ |x| \leq t\}, & \text{otherwise.} \end{cases} \tag{3.17}$$

Notice that $V_a g$ vanishes outside some bounded set, by Huygens' principle. Let $\chi \in C_0^\infty(B(0,a-1))$ be equal to 1 near $\overline{B(0,R_0)}$. (We will take $\chi \prec \chi_{a-1}$.) When $n \geq 3$ is odd, the corrected truncated solution of the wave equation (associated to P) is then by definition:

$$\widetilde{U}_a(t) = \psi_a(t)U(t)\chi_{a-1} - (1-\widetilde{\chi})(E_0*)(1-\chi)V_a(t), \tag{3.18}$$

where E_0* is the operator of convolution with E_0, and $\widetilde{\chi} \in C_0^\infty$ with $1_{\overline{B(0,R_0)}} \prec \widetilde{\chi} \prec \chi$. We get

$$(P+\partial_t^2)\widetilde{U}_a(t)g = \chi V_a(t)g + [-\Delta, \widetilde{\chi}](E_0*)(1-\chi)V_a(t)g. \tag{3.19}$$

The second term has uniformly compact support by the strong Huygens principle. Let us consider also

$$\widehat{U}_a(t)g := \psi_a(t)U(t)\chi_{a-1}g - (E_0*)(1-\chi)V_a(t)g \tag{3.20}$$

as a distribution in $(\mathbf{R}\times(\mathbf{R}^n \setminus \overline{B(0,R_0)})) \cup (]T_a+1,\infty[\times\mathbf{R}^n)$, so that

$$(\partial_t^2 - \Delta)\widehat{U}_a(t)g \in C^\infty(\mathbf{R}\times(\mathbf{R}^n \setminus B(0,R_0))) \cup (]T_a+1,\infty[\times\mathbf{R}^n)) \tag{3.21}$$

Now we shall use a *fact* about propagation of singularities for solutions of the wave equation. Let $\Omega \subset \mathbf{R}\times\mathbf{R}^n$ be open, $u \in \mathcal{D}'(\Omega)$, $(\partial_t^2 - \Delta)u \in C^\infty(\Omega)$:

If $(t_0, x_0) \in \operatorname{sing\,supp}(u)$ then there exists a maximally extended light ray $I \ni t \mapsto (t_0 + t, x_0 + t\omega)$ in Ω, where I is an open interval containing 0, $\omega \in S^{n-1}$, which passes through (t_0, x_0) and is contained in $\operatorname{sing\,supp} u$.

Let $(t_0, x_0) \in \operatorname{sing\,supp}(\widehat{U}_a(t)g)$, with $R_0 < |x_0| < a - 1$, $t_0 > T_a$. For $t > T_a$, $r_0 < |x| < a$, the term $\psi_a(t)U(t)\chi_{a-1}g$ is in C^∞, so (t_0, x_0) must belong to $\operatorname{sing\,supp} E_0 {*} (1-\chi)V_a(t)g \subset \{(t, x); (a-|x|)_+ \leq t - T_a\}$. We deduce that $t_0 > T_a + 1$. Also from the fact above, we see that there is a maximally extended ray in $(\mathbf{R} \times (\mathbf{R}^n \setminus B(0, R_0))) \cup (]T_a + 1, \infty[\times \mathbf{R}^n)$ passing through (t_0, x_0), contained in $\operatorname{sing\,supp}(\widehat{U}_a(t)g)$, which is well defined at $t = T_a + 1$ and with a corresponding point $(T_a + 1, x(T_a + 1))$ with $|x(T_a + 1)| \geq a - 1$. But then it is clear that the ray can be further extended backwards for all times $t < T_a + 1$ and the corresponding $x(t)$ stays outside $B(0, a - 1)$. This ray must be in $\operatorname{sing\,supp} \widehat{U}_a(t)g$ and we reach a contradiction for $|t|$ close to 0, since we are then outside the support of $\widehat{U}_a(t)g$.

In conclusion: $\widehat{U}_a(t)g$ and $(E_0{*})(1 - \chi)V_a(t)g$ are C^∞ for $t > T_a$, $R_0 < |x| < a - 1$. This implies that the last term in (3.19) is C^∞ for $|x| < a - 1$. Moreover, by the strong Huygens principle, it has compact support. So

$$(P + \partial_t^2)\widetilde{U}_a(t)g = W_a(t)g, \tag{3.22}$$

where $W_a(t)g \in C_0^\infty(]0, \infty[; \mathcal{D}^N_{\mathrm{comp}})$ has its support in some fixed compact set contained in $]T_a, \widehat{T}_a[\times \operatorname{supp}\chi$. (When n is even, we loose the compact support property in t.)

By the closed graph theorem,

$$W_a \in C_0^\infty(]0, \infty[; \mathcal{L}(\mathcal{H}, \mathcal{D}^N_{\mathrm{comp}})). \tag{3.23}$$

We also have the Cauchy data,

$$\widetilde{U}_a(t)g = 0,\ (\partial_t\widetilde{U}_a)(0)g = \chi_{a-1}g. \tag{3.24}$$

Finally $\widetilde{U}_a(t)$ has the same general regularity properties as U_a:

$$\widetilde{U}_a(t) \in C^m(\mathbf{R}_t; \mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j+\frac{1-m}{2}})),\ m \geq 0,\ j \geq 0. \tag{3.25}$$

(From (3.22), (3.24), we deduce that $\widetilde{U}_a - U\chi_{a-1} \in C^\infty(\mathbf{R}; \mathcal{L}(\mathcal{H}, \mathcal{D}^j))$, for all $j \geq 0$.)

When $\operatorname{Im}\lambda \geq C \gg 0$, we express $R(\lambda) = (P - \lambda^2)^{-1}$ by the integral formula (cf. (3.7)):

$$R(\lambda) = \int_0^\infty e^{it\lambda} U(t) dt. \tag{3.26}$$

In fact,

$$
\begin{aligned}
-\lambda^2 R(\lambda) &= \int_0^\infty \partial_t^2(e^{it\lambda})U(t)dt \qquad (3.27)\\
= -\int_0^\infty \partial_t(e^{it\lambda})\partial_t U(t)dt &= \partial_t U(0) + \int_0^\infty e^{it\lambda}\partial_t^2 U(t)dt\\
&= 1 + \int_0^\infty e^{it\lambda}\partial_t^2 U(t)dt,
\end{aligned}
$$

and it follows that

$$(P-\lambda^2)\int_0^\infty e^{it\lambda}U(t)dt = 1.$$

Put

$$R_a(\lambda) = \int_0^\infty e^{it\lambda}\widetilde{U}_a(t)dt : \mathcal{H} \to \mathcal{D}_{\mathrm{loc}}, \tag{3.28}$$

and notice that this operator is well defined and holomorphic for $\lambda \in \mathbf{C}$, since by the strong Huygens principle,

$$(\widetilde{U}_a(t)g)(x) = 0,\ \text{ for } t \geq \widetilde{T}_a + |x|. \tag{3.29}$$

The same calculation gives

$$(P-\lambda^2)R_a(\lambda) = \chi_{a-1} + \int_0^\infty e^{it\lambda}W_a(t)dt. \tag{3.30}$$

Since $\widetilde{\chi} \prec \chi \prec \chi_{a-1}$, the x-space projection of $\operatorname{supp} W_a(t)$ is contained in the region where $\chi_{a-1} = 1$. Writing

$$S_a(\lambda) = \int_0^\infty e^{it\lambda}W_a(t)g dt, \tag{3.31}$$

we get

$$(P-\lambda^2)R_a(\lambda) = \chi_{a-1}(1+S_a(\lambda)). \tag{3.32}$$

Lemma 3.3 *For every $N > 0$, $1 + S_a(\lambda) : \mathcal{H} \to \mathcal{H}$, $\mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{comp}}$ is invertible with inverse $\mathcal{O}(1)$ in $\mathcal{L}(\mathcal{H},\mathcal{H})$, for*

$$\operatorname{Im}\lambda \geq -N\log|\lambda|,\ \ |\lambda| \geq C(N).$$

Proof. We use the compact support property and (3.23) for W_a, and integrate by parts in (3.31):

$$S_a(\lambda) = \frac{1}{(i\lambda)^M}\int_0^\infty e^{it\lambda}(-\partial_t)^M W_a(t)dt.$$

The $\mathcal{H} \to \mathcal{H}$ norm of this operator is $\leq C_M|\lambda|^{-M}e^{\widehat{T}_a(\operatorname{Im}\lambda)_-}$, with the standard notation $b = b_+ - b_-$, $b_\pm = \max(\pm b, 0)$. If $\operatorname{Im}\lambda \geq -N\log|\lambda|$, we get

$$\|S_a(\lambda)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq C_M|\lambda|^{\widehat{T}_a N - M},$$

which is $\leq \frac{1}{2}$ if we first take $M > \widehat{T}_a N$, and then $|\lambda|$ large enough. Notice that $(1+S_a(\lambda))^{-1} = 1 - S_a(\lambda)(1+S_a(\lambda))^{-1}$ maps $\mathcal{H}_{\text{comp}} \to \mathcal{H}_{\text{comp}}$. #

When $\operatorname{Im}\lambda \geq C \gg 0$, we get from (3.32) the relation

$$R(\lambda)\chi_{a-1} = R_a(\lambda)(1+S_a(\lambda))^{-1}, \tag{3.33}$$

for $R(\lambda) = (P-\lambda^2)^{-1}$, and we can use this relation to get a meromorphic extension of $R(\lambda)\chi_{a-1}$ to $\mathbf{C}$, (since $S_a(\lambda)$ is compact and holomorphic in λ). If (as before), we let $R(\lambda)$ denote also the meromorphic extension of $R(\lambda) : \mathcal{H}_{\text{comp}} \to \mathcal{H}_{\text{loc}}$, then (3.33) extends to $\lambda \in \mathbf{C}$, and we conclude that $R(\lambda)\chi_{a-1}$ has at most finitely many poles in $\operatorname{Im}\lambda \geq -N\log_+|\lambda|$, for every $N \geq 0$. Applying Proposition 2.3, we then get the following result (which remains true for n even).

Proposition 3.4 *In addition to the general bb assumptions of chapter 2, we assume that $P \geq -C$ and that the non-trapping assumption (3.13) holds. We also assume that $n \geq 3$ is odd. Let $R(\lambda)$ be the meromorphic extension of $(P-\lambda^2)^{-1} : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}$. Then for every $N \geq 0$, $R(\lambda)$ has at most finitely many poles in* $\operatorname{Im}\lambda \geq -N\log_+|\lambda|$.

We next estimate the truncated extension of $R(\lambda)$ and start with R_a. If $\chi \in C_0^\infty(\mathbf{R}^n)$ is equal to 1 near $\overline{B(0,R_0)}$, then (by the strong Huygens principle) $\chi\widetilde{U}_a(t)$ vanishes for $t \geq T_\chi$ for some sufficiently large $T_\chi > 0$. Hence

$$\chi R_a(\lambda) = \int_0^{T_\chi} e^{it\lambda}\chi\widetilde{U}_a(t)dt, \tag{3.34}$$

and using (3.25), we get after an integration by parts:

$$\|\chi R_a(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^j)} \leq \frac{C_j}{|\lambda|}e^{T_\chi(\operatorname{Im}\lambda)_-},\ j \geq 0. \tag{3.35}$$

Combining this with (3.33) and Lemma 3.3, we get

$$\|\chi R(\lambda)\chi_{a-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \frac{C}{|\lambda|}e^{T_\chi(\operatorname{Im}\lambda)_-}, \tag{3.36}$$

for $|\lambda| \geq C(N)$, $\mathrm{Im}\,\lambda \geq -N\log_+|\lambda|$. Here we will need to improve the exponent -1 in $|\lambda|^{-1}$ by replacing $\|\cdot\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$ by $\|\cdot\|_{\mathcal{L}(\mathcal{D}^\theta,\mathcal{H})}$

As a preparation for the promised asymptotic expansion of the wave group for $t \to +\infty$, we recall that

$$R(\lambda) = \int_0^\infty e^{it\lambda}U(t)dt,\ \mathrm{Im}\,\lambda \gg 0,$$

and we shall establish a corresponding "Fourier inversion" formula, which could be obtained by selfadjoint functional calculus, but which we prefer to derive by direct arguments. For this, we work in a region $\mathrm{Im}\,\lambda \geq C \gg 0$ and start by getting some further estimates on $R(\lambda)$.

First, by (3.7), (3.26), we get

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^{j+\frac{1}{2}})} \leq \mathcal{O}(1), \tag{3.37}$$

and after integration by parts in (3.26),

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^j)} \leq \frac{\mathcal{O}(1)}{\langle\lambda\rangle}. \tag{3.38}$$

From the first of the two equivalent identities

$$PR(\lambda) = 1 + \lambda^2 R(\lambda),\ R(\lambda) = -\frac{1}{\lambda^2} + \frac{1}{\lambda^2}PR(\lambda), \tag{3.39}$$

and (3.38), we get

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^{j+1})} \leq \mathcal{O}(1)\langle\lambda\rangle. \tag{3.40}$$

Use (3.39), (3.37) to get

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^{j-1/2})} \leq \frac{\mathcal{O}(1)}{\langle\lambda\rangle^2}. \tag{3.41}$$

The last two estimates are the most extreme ones, and (3.37), (3.38) can be obtained from these two by interpolation (cf (3.2)) and we get more generally

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{D}^j,\mathcal{D}^{j-\alpha})} \leq \frac{\mathcal{O}(1)}{\langle\lambda\rangle^{1+2\alpha}},\ -1 \leq \alpha \leq \frac{1}{2}. \tag{3.42}$$

We claim that $U(t) = \widehat{U}(t)$, where

$$\widehat{U}(t) = \frac{1}{2\pi}\int_{\mathrm{Im}\,=C} e^{-it\lambda}R(\lambda)d\lambda,\ t \geq 0. \tag{3.43}$$

The integral converges in $\mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j-\epsilon})$ for every $\epsilon > 0$ and becomes a continuous function of t with values in that space. Using (3.39), we get

$$\begin{aligned}\widehat{U}(t)g &= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C} e^{-it\lambda}(-\frac{1}{\lambda^2}+\frac{R(\lambda)}{\lambda^2}P)gd\lambda \\ &= \frac{1}{2\pi}\int_\gamma e^{-it\lambda}\frac{1}{\lambda^2}d\lambda\, g + \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C} e^{-it\lambda}\frac{R(\lambda)}{\lambda^2}Pgd\lambda \\ &= tg + \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C} e^{-it\lambda}\frac{R(\lambda)}{\lambda^2}Pgd\lambda,\end{aligned}$$

where γ is a small positively oriented circle around 0.

In the last integral, we gained a sufficiently large negative power of λ, to be able to apply ∂^2 and get

$$\partial_t^2\widehat{U}(t)g = -\frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C} e^{-it\lambda}R(\lambda)Pgd\lambda = -P\widehat{U}(t)g,$$

i.e.

$$(P+\partial_t^2)\widehat{U}(t)g = 0. \tag{3.44}$$

Further,

$$\widehat{U}(0)g = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C}\frac{R(\lambda)}{\lambda^2}Pgd\lambda = 0,$$

by contour deformation upwards, and

$$\partial_t\widehat{U}(0)g = g - \frac{i}{2\pi}\int_{\operatorname{Im}\lambda=C}\frac{R(\lambda)}{\lambda}Pgd\lambda = g,$$

by the same contour deformation.

We have then verified that $\widehat{U}(t)$ solves the same Cauchy problem as U, so we have $U = \widehat{U}$:

$$U(t) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=C} e^{-it\lambda}R(\lambda)d\lambda,\ t \geq 0, \tag{3.45}$$

where the integral converges in $\mathcal{L}(\mathcal{D}^j, \mathcal{D}^{j-\epsilon})$ for every $\epsilon > 0$.

We want to multiply this to the left by χ and to the right by χ_{a-1}, and then make contour deformation, and use an estimate like (3.36), but with a little faster decay than $|\lambda|^{-1}$, so we start by obtaining such improved decay. For $g \in \mathcal{D}$ we write

$$\chi R(\lambda)\chi_{a-1}g = \chi(-\frac{1}{\lambda^2}+\frac{R(\lambda)}{\lambda^2}\widetilde{\chi}_{a-1}P)\chi_{a-1}g,$$

where $\chi_{a-1} \prec \widetilde{\chi}_{a-1} \in C_0^\infty(B(0,a-1))$. Using (3.36) (with χ_{a-1} there replaced by $\widetilde{\chi}_{a-1}$) we get for the same λ as there,

$$\|\chi R(\lambda)\chi_{a-1}\|_{\mathcal{L}(\mathcal{D},\mathcal{H})} \leq \frac{C}{|\lambda|^2} e^{T_\chi (\operatorname{Im}\lambda)_-}. \tag{3.46}$$

(A more optimal estimate would probably be to have $\mathcal{L}(\mathcal{D}^{1/2},\mathcal{H})$ instead of $\mathcal{L}(\mathcal{D},\mathcal{H})$.) Interpolating between (3.36) and (3.46), we get

$$\|\chi R(\lambda)\chi_{a-1}\|_{\mathcal{L}(\mathcal{D}^\theta,\mathcal{H})} \leq \frac{C}{|\lambda|^{1+\theta}} e^{T_\chi (\operatorname{Im}\lambda)_-}, \tag{3.47}$$

for λ as in (3.36) and $0 \leq \theta \leq 1$.

With this in mind, we write

$$\chi U(t)\chi_{a-1} g = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = C} e^{-it\lambda}\chi R(\lambda)\chi_{a-1} d\lambda \tag{3.48}$$
$$= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=-A} e^{-it\lambda}\chi R(\lambda)\chi_{a-1} d\lambda + \sum_{\operatorname{Im}\lambda > -A} \text{residues},$$

where we choose $A > 0$ such that no resonances has the imaginary part $-A$. The $\mathcal{L}(\mathcal{D}^\theta,\mathcal{H})$-norm of the integral in the last expression in (3.48) can be estimated by

$$\mathcal{O}_A(1)e^{-tA+T_\chi A} = \mathcal{O}_A(1)e^{-At}.$$

Let $\lambda_0 \in \mathbf{C}$, be a pole of the meromorphic extension of $(P-\lambda^2)^{-1}$, so that in the case $\lambda_0 \neq 0$ (cf (2.48)) we have near λ_0:

$$-R(\lambda) = \sum_1^{\omega(\lambda_0)} \frac{\Pi_{-j}}{(\lambda^2-\lambda_0^2)^j} + \operatorname{Hol}(\lambda) \tag{3.49}$$

In the case $\operatorname{Im}\lambda_0 \leq 0$ we discussed some of the structure of the "spectral projections" Π_{-j}. When $\operatorname{Im}\lambda_0 > 0$ we know that λ_0 is purely imaginary and that λ_0^2 is a negative eigenvalue of P. In this case Π_{-1} is the ordinary orthogonal spectral projection, while $\Pi_{-j} = 0$ for $j \geq 2$. In the case $\lambda_0 = 0$ things are a little more complicated and we make no further comments in that case. For $\lambda_0 \neq 0$ the corresponding contribution to the sum of residues in (3.48) is

$$\sum_1^{\omega(\lambda_0)} \frac{1}{2\pi}\int_{\gamma_0} e^{-it\lambda}\frac{1}{(\lambda^2-\lambda_0^2)^j} d\lambda\, \chi\Pi_{-j}\chi_{a-1}, \tag{3.50}$$

where γ_0 is a small positively oriented circle around λ_0. The integral is of the form $e^{-it\lambda_0}$ times a polynomial of degree at most $j-1$ in t. For $j=1$, we get

$$\frac{1}{2\pi}2\pi i\frac{e^{-it\lambda_0}}{2\lambda_0}\chi\Pi_{-1}\chi_{a-1}=\frac{ie^{-it\lambda_0}}{2\lambda_0}\chi\Pi_{-1}\chi_{a-1}. \tag{3.51}$$

We have proved the following result essentially due to Lax and Phillips with a more straight forward proof due to Vainberg [91], extended to the black box framework by Tang-Zworski [90], that we have modified a little to decrease the number of cut-offs:

Theorem 3.5 *Let $n\ge 3$ be odd. Let P be a black box operator as in chapter 2 and assume (3.1), (3.13). Then for every $A>0$ such that $-A$ is not the imaginary part of any resonance, and for every $\chi\in C_0^\infty(\mathbf{R}^n)$ equal to a constant near $\overline{B(0,R_0)}$, we have*

$$\chi U(t)\chi=\sum_{\lambda\in\mathrm{Res}\,(P)\cup\{i\mu;\mu>0,\,-\mu^2\in\sigma(P))\}}e^{-it\lambda}\chi p_{\lambda,P}(t)\chi+r_{\chi,A}(t),$$

where $p_{\lambda,P}(t)$ is a polynomial in t with values in the operators $\mathcal{H}_{\mathrm{comp}}\to\mathcal{D}_{\mathrm{loc}}$. When λ_0 is a simple pole of the resolvent and $\lambda_0\neq 0$, we have $p_{\lambda,P}(t)=\frac{i}{2\lambda}\Pi_{-1,\lambda}$, where $\Pi_{-1,\lambda}$ is the corresponding spectral projection. The remainder term satisfies $\|r_{\chi,A}(t)\|_{\mathcal{L}(\mathcal{D}^\theta,\mathcal{H})}\le C_{\chi,A,\theta}e^{-tA}$, for every $\theta>0$.

According to Tang-Zworski, Burq, there is a trick of C. Morawetz, which allows us to replace $\mathcal{D}^\theta$ by $\mathcal{H}$ in the above theorem.

4 Absence of resonances exponentially close to R.

In this chapter we shall descibe a recent result of N. Burq [15] which says that under quite general assumptions there are no resonances exponentially close to the real axis. We start by formulating the precise theorem. Let $\mathcal{O}\subset\subset\mathbf{R}^n$ be an open set with a smooth (C^∞) boundary $\partial\mathcal{O}$ and assume that $\mathbf{R}^n\setminus\mathcal{O}$ is connected. Let

$$P=\sum_{|\alpha|\le 2}a_\alpha(x)D^\alpha \tag{4.1}$$

be an elliptic formally selfadjoint operator with smooth coefficients on $\mathbf{R}^n\setminus\mathcal{O}$. We assume that

$$P=-\Delta\text{ for }|x|>R_0-1, \tag{4.2}$$

for some sufficiently large $R_0 > 0$, that we choose large enough so that $\overline{\mathcal{O}} \subset B(0, R_0 - 1)$.

We explain the main ideas of the proof of the following result of N. Burq [15].

Theorem 4.1 *Let P also denote the Dirichlet realization of P in $L^2(\mathbf{R}^n \backslash \mathcal{O})$. Then there exists a constant $C > 0$, such that P has no resonances in the set of $\lambda \in \mathbf{C}$ such that*

$$|\lambda| \geq C, \ \operatorname{Im} \lambda \geq -e^{-C|\operatorname{Re} \lambda|}. \tag{4.3}$$

Actually, Burq also establishes this result for the Neumann realization of P and even in the more general case when P is realized with Dirichlet conditions on one connected part of the boundary and with Neumann conditions on the other part. As a corollary, Burq shows that the local energy of solutions to the wave equation with compactly supported initial data decays at least as fast as $(\log(1+t))^{-k}$, $t \to +\infty$, where $k \geq 0$ depends on the smoothness of the Cauchy data at $t = 0$. Here smoothness of the data is measured in terms of being in the domain of a certain power of the generator of the wave group, when the latter is considered as a matrix solution to a first order system. Earlier results in this direction were obtained by E. Harrell [36] and C. Fernandez and R. Lavine [24]. After the work [15] there have been extensions by G. Vodev [97, 98], and by Burq [16]. Vodev observed some simplifications in Burq's original proof, and we shall use one of them in our presentation, as well as a new estimate (that will not be explained in detail here) that is planned to appear in a joint work with Burq. This estimate permits to give a treatment near infinity which is more natural and closer to the intuition.

The first step in the proof is to notice that it suffices to show that for real λ with $|\lambda| \geq 1$, we have

$$\|R(\lambda)v\|_{H^1(B(0,R_0)\backslash\mathcal{O})} \leq C_0 e^{C_0|\lambda|} \|v\|_{L^2}, \tag{4.4}$$

for $v \in L^2_{\mathrm{comp}}(B(0, R_0) \setminus \mathcal{O})$ for some $C_0 > 0$. In fact, assuming (4.4), we shall extend $R(\mu)$ holomorphically to μ in an exponentially small disc centered at λ as an operator $L^2_{\mathrm{comp}}(\mathbf{R}^n \setminus \mathcal{O}) \to (H^2 \cap H^1_0)_{\mathrm{loc}}(\mathbf{R}^n \setminus \mathcal{O})$. Let $1_{B(0,R_0-1)} \prec \chi_0 \prec \chi_1 \prec \chi_2 \prec 1_{B(0,R_0)}$, with $\chi_j \in C_0^\infty$. As an approximation for $R(\mu)\chi_1$, we take

$$A(\mu) := \chi_2 R(\lambda)\chi_1 - R_0(\mu)[P_0, \chi_2]R(\lambda)\chi_1.$$

Then

$$(P-\mu^2)A(\mu) = \tag{4.5}$$
$$\chi_1 + [P_0,\chi_2]R(\lambda)\chi_1 + \chi_2(\lambda^2-\mu^2)R(\lambda)\chi_1 - [P_0,\chi_2]R(\lambda)\chi_1$$
$$= \chi_1 + \chi_2(\lambda^2-\mu^2)R(\lambda)\chi_1.$$

For the exterior part of the approximate resolvent, we first compute

$$(P-\mu^2)(1-\chi_0)R_0(\mu)(1-\chi_1) = 1-\chi_1 - [P_0,\chi_0]R_0(\mu)(1-\chi_1).$$

Correcting for the last term, we are led to the approximation of $R(\mu)(1-\chi_1)$:

$$B(\mu) := (1-\chi_0)R_0(\mu)(1-\chi_1) + A(\mu)[P_0,\chi_0]R_0(\mu)(1-\chi_1).$$

Then

$$\begin{aligned}(P-\mu^2)B(\mu) = \ & 1-\chi_1-[P_0,\chi_0]R_0(\mu)(1-\chi_1)\\ & +(\chi_1+\chi_2(\lambda^2-\mu^2)R(\lambda)\chi_1)[P_0,\chi_0]R_0(\mu)(1-\chi_1)\\ = \ & (1-\chi_1)+\chi_2(\lambda^2-\mu^2)R(\lambda)[P_0,\chi_0]R_0(\mu)(1-\chi_1).\end{aligned}$$

Put $\widetilde{R}(\mu) = A(\mu) + B(\mu)$. Then

$$(P-\mu^2)\widetilde{R}(\mu) = 1 + K, \tag{4.6}$$

$$K = \chi_2(\lambda^2-\mu^2)R(\lambda)(\chi_1 + [P_0,\chi_0]R_0(\mu)(1-\chi_1)). \tag{4.7}$$

Using the Fourier transform and complex deformation of integration contours as in section 2.1, we see that

$$[P_0,\chi_0]R_0(\mu) : L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O}) \to L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O})$$

is of norm $\leq Ce^{C|\mu|}$. It follows that

$$K : L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O}) \to L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O})$$

and has operator norm $\mathcal{O}(1)|\lambda|e^{C_0|\lambda|}|\lambda-\mu|$, so if $|\lambda-\mu| \leq e^{-2C_0|\lambda|}$ and $|\lambda| \gg 1$, $1+K$ has a bounded inverse and we get the holomorphic extension

$$R(\mu) = \widetilde{R}(\mu)(1+K)^{-1} : L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O}) \to (H^2\cap H^1_0)_{\mathrm{loc}}(B(0,R_0)\setminus\mathcal{O}). \tag{4.8}$$

Proposition 2.3 then shows that $R(\mu)$ extends holomorphically also as an operator on $L^2_{\rm comp}(\mathbf{R}^n \setminus \mathcal{O})$.

One of the main steps of the proof is to establish weighted L^2 estimates in a bounded domain for $(P - \lambda^2) = \lambda^2(h^2P - 1)$, with $h = 1/|\lambda|$. These estimates are so called Carleman estimes, developed with enormous success by Hörmander and others in different branches of partial differential equations, including uniqueness problems and Cauchy-Riemann type equations. In the context of spectral problems, they have been used by G. Lebeau and L. Robbiano [52, 53]. Much works in a more general frame work, so to start with, we take a more general semi-classical operator.

4.1 Local Carleman estimates away from the boundary.

Let $\Omega \subset \mathbf{R}^n$ be some open set. Let P be a semiclassical differential operator of order 2, formally self-adjoint and of the form:

$$P = \sum_{|\alpha|\le 2} a_\alpha(x;h)(hD_x)^\alpha, \ a_\alpha(x;h) \in C^\infty(\Omega). \tag{4.9}$$

Assume that

$$a_\alpha(x;h) = a^0_\alpha(x) + \mathcal{O}(h) \text{ in } C^\infty(\Omega), \tag{4.10}$$

and that a_α is independent of h for $|\alpha| = 2$. The semiclassical principal symbol of P is then by definition:

$$p(x,\xi) = \sum_{|\alpha|\le 2} a^0_\alpha(x)\xi^\alpha = a(x,\xi) + \sum_{|\alpha|\le 1} a^0_\alpha(x)\xi^\alpha. \tag{4.11}$$

Here $a(x,\xi) = \sum_{|\alpha|=2}(...)$ is equal to the ordinary principal symbol of the operator $h^{-2}P$. We assume that P is elliptic in the classical sense:

$$a(x,\xi) \ge \frac{1}{C}|\xi|^2. \tag{4.12}$$

Let $\phi \in C^\infty(\Omega;\mathbf{R})$ and consider

$$P_\phi = e^{\phi(x)/h} P e^{-\phi(x)/h}. \tag{4.13}$$

This operator is also of the form (4.9) with the same ordinary principal symbol $h^2 a(x,\xi)$ and with the new semiclassical principal symbol

$$p_\phi(x,\xi) = p(x, \xi + i\phi'(x)). \tag{4.14}$$

Assume that we have

$$p_\phi(x,\xi)=0 \Rightarrow \frac{1}{i}\{p_\phi,\overline{p_\phi}\}<0, \tag{4.15}$$

where

$$\{f,g\}=\sum_{j=1}^{n}(\frac{\partial f}{\partial \xi_j}(x,\xi)\frac{\partial g}{\partial x_j}(x,\xi)-\frac{\partial f}{\partial x_j}(x,\xi)\frac{\partial g}{\partial \xi_j}(x,\xi)),$$

denotes the Poisson bracket of two differentiable functions f,g. Notice that $\frac{1}{i}\{f,\overline{f}\}=-2\{\mathrm{Re}\, f,\mathrm{Im}\, f\}$ is a real-valued function.

The basic Carleman estimate is then given by:

Lemma 4.2 *Let $W\subset\subset\Omega$ be open. Then there exist constants $C_0,h_0>0$ such that for all $0<h\le h_0$, $u\in C_0^\infty(W)$:*

$$h^{1/2}\|u\|_{H^2}\le C_0\|P_\phi u\|. \tag{4.16}$$

Here we let H^s denote the semi-classical Sobolev space equipped with the h-dependent norm $\|u\|_{H^s}=\|\langle hD\rangle^s u\|$.

We will only give an outline of the **proof**, mainly for readers who are familiar with pseudodifferential operator machinery especially in the semi-classical setting. However, we point out that some basic ideas are older than the theory of pseudodifferential operators. Start by looking at the trivial identity:

$$\begin{aligned}\frac{1}{h}\|P_\phi u\|^2 &\ge \frac{1}{h}\|P_\phi u\|^2-\frac{1}{h}\|P_\phi^* u\|^2 \\ &=\frac{1}{h}((P_\phi^*P_\phi-P_\phi P_\phi^*)u|u)=(\frac{1}{h}[P_\phi^*,P_\phi]u|u).\end{aligned} \tag{4.17}$$

Here the commutator has the form

$$\frac{1}{h}[P_\phi^*,P_\phi]=\sum_{|\alpha|\le 3}b_\alpha(x;h)(hD_x)^\alpha,$$

with

$$b_\alpha(x;h)=b_\alpha^0(x)+\mathcal{O}(h) \text{ in } C^\infty(\Omega),$$

and the associated semi-classical principal symbol is

$$\frac{1}{i}\{\overline{p_\phi},p_\phi\}$$

which is > 0 on $p_\phi^{-1}(0)$. Since $|p_\phi(x,\xi)| \sim \langle\xi\rangle^2$ for large ξ, we can find constants $b, \widetilde{C} > 0$, such that

$$\frac{1}{i}\{\overline{p_\phi}, p_\phi\} + b|p_\phi(x,\xi)|^2 \geq \frac{1}{\widetilde{C}}\langle\xi\rangle^4 \text{ on } \widetilde{W}\times\mathbf{R}^n,$$

where we choose $\widetilde{W} \subset\subset \Omega$ open with $W \subset\subset \widetilde{W}$.

Using h-pseudodifferential calculus and/or a suitable version of Gårding's inequility (see for instance [21]), we deduce that

$$(\frac{1}{h}[P_\phi^*, P_\phi]u|u) + b\|P_\phi u\|^2 \geq \frac{1}{2\widetilde{C}}\|u\|_{H^2}^2,\ u \in C_0^\infty(W),$$

We get

$$\|u\|_{H^2}^2 \leq 2\widetilde{C}((\frac{1}{h}[P_\phi^*, P_\phi]u|u) + b\|P_\phi u\|^2). \tag{4.18}$$

Combining this with (4.17), we get (4.16). #

Actually, we shall not use the full strength of (4.16) but only the weaker estimate

$$h^{1/2}\|u\|_{H^1} \leq \mathcal{O}_W(1)\|P_\phi u\|,\ u \in C_0^\infty(W). \tag{4.19}$$

Taking squares and recalling the definition of P_ϕ, we get

$$h\int(e^{2\phi/h}|u|^2 + |h\nabla_x(e^{\phi/h}u)|^2)dx \leq \mathcal{O}_W(1)\int e^{2\phi/h}|Pu|^2dx. \tag{4.20}$$

This can also be written:

$$h\int e^{2\phi/h}(|u|^2 + |h\nabla_x u|^2)dx \leq \mathcal{O}_W(1)\int e^{2\phi/h}|Pu|^2dx. \tag{4.21}$$

4.2 Local Carleman estimates at the boundary

Now let $\Omega \subset \mathbf{R}^n$ be closed, bounded and have a smooth boundary $\partial\Omega$, Let $x_0 \in \partial\Omega$ and assume that P, ϕ are as above in a neighborhood W (in $\mathbf{R}^n$) of x_0. We want to have (4.21) for $u \in C_0^\infty(\Omega\cap W)$ with $u_{|\partial\Omega} = 0$. Using the ellipticity, it is quite classical that we can find local coordinates $x_1, .., x_n$ centered at x_0, such that Ω is defined in W by $x_n \geq 0$, and the (new) principal symbol becomes

$$a(x,\xi) = \xi_n^2 + r(x,\xi'),\ \xi' = (\xi_1, .., \xi_{n-1}). \tag{4.22}$$

So

$$P = (hD_{x_n})^2 + R(x, hD_{x'}; h) + b(x; h)hD_{x_n}. \tag{4.23}$$

The Lebesgue measure becomes $\lambda(x)dx$ for some smooth and strictly positive function $\lambda(x)$. Define P_ϕ as above. It is now practical to organize the computations a little differently, so we write

$$P_\phi = Q_2 + iQ_1,\ Q_2 = \frac{1}{2}(P_\phi + P_\phi^*),\ Q_1 = \frac{1}{2i}(P_\phi - P_\phi^*).$$

Here the * indicates that the we take the formal adjoint in $L^2(W; \lambda(x)dx_1..dx_n)$ and norms and scalar products will be in this space for a while. The subscript j in Q_j indicates that we have a differential operator of order j.

Then for all $u \in C_0^\infty(\Omega \cap W)$ with $u_{|\partial\Omega} = 0$, we get

$$\|P_\phi u\|^2 = \|Q_2 u\|^2 + \|Q_1 u\|^2 + i\Big[(Q_1 u|Q_2 u) - (Q_2 u|Q_1 u)\Big]. \tag{4.24}$$

If we could move the Q_j from the right to the left in each of the two scalar products, the bracket in (4.24) would become

$$(i[Q_2, Q_1]u|u) = \frac{1}{2}([P_\phi^*, P_\phi]u|u),$$

and we can hope to estimate this from below as before. (The presence of the factor 1/2 is explained by the fact that $\|Q_2 u\|^2 + \|Q_1 u\|^2$ is not in general equal to $\|P_\phi^* u\|^2$, but rather to $\frac{1}{2}\|P_\phi u\|^2 + \frac{1}{2}\|P_\phi^* u\|^2$.) However, we have to take into account the boundary terms, when carrying out the integrations by parts that pull over the Q_j to the left. Because of the Dirichlet condition, we get no boundary integral, when moving over Q_1, while Q_2 will give rise to a boundary term from its $(hD_{x_n})^2$ term. Notice that $Q_2 = (hD_{x_n})^2$ plus terms which contain hD_{x_n} at most to the power 1. Hence,

$$i(Q_1 u|Q_2 u) = i(Q_2 Q_1 u|u) + h\int_{x_n=0} (Q_1 u)\overline{hD_{x_n} u}\lambda(x', 0)dx'. \tag{4.25}$$

Only the hD_{x_n} term in Q_1 can give a non-vanishing contribution to the boundary integral (because of the Dirichlet condition) so we compute this term, using (4.23):

$$\begin{aligned} P_\phi &= (hD_{x_n} + i\partial_{x_n}\phi)^2 + R_\phi(x, hD_{x'}; h) + b(x; h)(hD_{x_n} + i\partial_{x_n}\phi) \\ &= (hD_{x_n})^2 - (\partial_{x_n}\phi)^2 + i((\partial_{x_n}\phi) \circ hD_{x_n} + (hD_{x_n}) \circ (\partial_{x_n}\phi)) + \\ &\quad R_\phi(x, hD_{x'}; h) + bhD_{x_n} + ib\partial_{x_n}\phi \\ &= P - (\partial_{x_n}\phi)^2 + (R_\phi - R) + \\ &\quad i((\partial_{x_n}\phi) \circ hD_{x_n} + (hD_{x_n}) \circ (\partial_{x_n}\phi)) + ib\partial_{x_n}\phi. \end{aligned}$$

Now use that $(hD_{x_n})^* = hD_{x_n} + h\mu(x)$ for some smooth function μ (which depends on λ) and

$$\begin{aligned} P_\phi^* &= P - (\partial_{x_n}\phi)^2 + (R_\phi^* - R^*) - i((hD_{x_n})^* \circ (\partial_{x_n}\phi) + (\partial_{x_n}\phi)(hD_{x_n})^*) \\ &\quad -ib\partial_{x_n}\phi, \end{aligned}$$

to conclude that the hD_{x_n} term in $P_\phi - P_\phi^*$ is $4i(\partial_{x_n}\phi)hD_{x_n}$, so

$$Q_1 = \frac{1}{2i}(P_\phi - P_\phi^*) = 2(\partial_{x_n}\phi)\, hD_{x_n} + \text{ a tangential operator,} \tag{4.26}$$

where by a tangential operator we mean one which contains differentiations with respect to $x_1, .., x_{n-1}$ but not with respect to x_n.

Now (4.25) becomes

$$i(Q_1 u|Q_2 u) = i(Q_2 Q_1 u|u) + 2h \int_{x_n=0} (\partial_{x_n}\phi)|hD_{x_n}u|^2 \lambda(x', 0)dx'. \tag{4.27}$$

Using this in (4.24), we get

$$\begin{aligned} \|P_\phi u\|^2 &= \|Q_2 u\|^2 + \|Q_1 u\|^2 + \\ &\quad 2h \int_{x_n=0} (\partial_{x_n}\phi)|hD_{x_n}u|^2 \lambda(x', 0)dx' + (i[Q_2, Q_1]u|u), \end{aligned} \tag{4.28}$$

so we can hope to have (4.21) also in the boundary case, provided that

$$\partial_{x_n}\phi_{|x_n=0} > 0. \tag{4.29}$$

A coordinate free formulation of the last condition is that

$$\frac{\partial \phi}{\partial \nu}_{|\partial\Omega} > 0, \tag{4.30}$$

where ν denotes the interior unit normal of $\partial\Omega$, for the metric given by the quadratic form on the tangent space which is dual to $a(x, \xi)$.

The treatment of the last term in (4.28) becomes more technical because of the boundary (and requires also the use of tangential pseudodifferential operators and a division trick going back to a work of V. Ivrii). This was carried out by Lebeau–Robbiano [52, 53] also in the case of Neumann boundary conditions. We will not go into the technical details and simply state the result, as it was formulated by Burq [15]: Let P be as above, but of a more restricted form:

$$P = \sum_{|\alpha|=2} a_\alpha(x)(hD_x)^\alpha + h \sum_{|\alpha|\le 1} b_\alpha(x; h)(hD_x)^\alpha - 1, \tag{4.31}$$

with $b_\alpha(x; h) = \mathcal{O}(1)$ in $C^\infty(\Omega)$. Then we have

Proposition 4.3 *Let P, ϕ, Ω be as above (including (4.30), (4.31)) and let W be a sufficiently small neighborhood of $x_0 \in \partial\Omega$. Then for $h > 0$ small enough, and all $u \in C_0^\infty(\Omega \cap W)$ with $u_{|\partial\Omega} = 0$, we have (4.20), (4.21), with $\Omega \cap W$ as the integration domain.*

4.3 Gluing local Carleman estimates for a given ϕ.

Let $\mathcal{O} \subset \mathbf{R}^n$ be closed with smooth boundary, and let $W \subset\subset \mathbf{R}^n$ be open. Let P be of the form (4.9) near $\overline{W}$ and let ϕ be smooth and real near $\overline{W}$. Assume that $\overline{W}$ is covered by finitely many open sets $W_1, .., W_N$ such that (4.20) (or equivalently (4.21)) holds for $u \in C_0^\infty(\Omega \cap W_j)$ with $u_{|\partial\Omega} = 0$. Then we shall see that it also holds for all $u \in C_0^\infty(\Omega \cap W)$ with $u_{|\partial\Omega} = 0$.

Indeed, let $0 \le \chi_j \in C_0^\infty(W_j)$ with $\sum_1^N \chi_j^2 = 1$ near $\overline{W}$ and apply (4.21) to $\chi_j u$:

$$\begin{aligned} h \int e^{2\phi/h}(|\chi_j u|^2 + |\chi_j h\nabla_x u|^2)dx \le& \\ \mathcal{O}(1) \int e^{2\phi/h}|\chi_j Pu|^2 dx &+ \mathcal{O}(1) \int e^{2\phi/h}|[P, \chi_j]u|^2 dx \\ &+ \mathcal{O}(1) \int e^{2\phi/h}|[h\nabla_x, \chi_j]u|^2 dx. \end{aligned} \tag{4.32}$$

Summing and using that $[P, \chi_j] = h \sum_{|\alpha|\le 1} b_{\alpha,j}(x; h) hD_x$, we get

$$\begin{aligned} h \int e^{2\phi/h}(|u|^2 + |h\nabla u|^2)^2 dx \;\le\; & \mathcal{O}(1) \int e^{2\phi/h}|Pu|^2 dx + \\ & \mathcal{O}(1)h^2 \int e^{2\phi/h}(|u|^2 + |h\nabla_x u|^2)dx. \end{aligned} \tag{4.33}$$

For h small enough, the last term can be absorbed by the first member of the inequality. #

4.4 Construction of ϕ satisfying the Poisson bracket condition.

Our function ϕ will satisfy

$$|d\phi| \ge \text{Const.} > 0. \tag{4.34}$$

Write

$$p(x,\xi)=a(x,\xi)+p_1(x,\xi)+p_0(x),\ a(x,\xi)=\langle A(x)\xi,\xi\rangle. \tag{4.35}$$

We will replace ϕ by $\tau\phi$ for $\tau>0$ large enough. Then

$$\begin{aligned} & p_{\tau\phi}(x,\xi)=p(x,\xi+i\tau\phi'_x)= \qquad (4.36)\\ & a(x,\xi)-\tau^2 a(x,\phi')+2i\tau\langle A(x)\xi,\phi'(x)\rangle+p_1(x,\xi)+i\tau p_1(x,\phi'(x))+p_0(x)\\ & \quad =\tau^2(\widetilde{q}_2(x,\xi)+2i\widetilde{q}_1(x,\xi)),\end{aligned}$$

with

$$\widetilde{q}_2(x,\xi)=a(x,\frac{\xi}{\tau})-a(x,\phi'_x)+\frac{1}{\tau}p_1(x,\frac{\xi}{\tau})+\frac{1}{\tau^2}p_0(x),$$

$$\widetilde{q}_1(x,\xi)=\langle A(x)\frac{\xi}{\tau},\phi'(x)\rangle+\frac{1}{2\tau}p_1(x,\phi'(x)).$$

Let us look for the commons zeros of the real and imaginary parts: First notice that at such a common zero, we must have $|\xi|\sim\tau|\phi'(x)|$, uniformly with respect to ϕ, for τ sufficiently large. Then we get

$$\begin{cases} a(x,\frac{\xi}{\tau})-a(x,\phi'(x)=\mathcal{O}_\phi(1/\tau)\\ \langle A(x)\frac{\xi}{\tau},\phi'(x)\rangle=\mathcal{O}_\phi(1/\tau).\end{cases} \tag{4.37}$$

At such common zeros, we compute

$$\begin{aligned}\frac{\partial\widetilde{q}_2}{\partial\xi} &= \frac{1}{\tau}2A(x)\frac{\xi}{\tau}+\mathcal{O}_\phi(\frac{1}{\tau^2}),\\ \frac{\partial\widetilde{q}_2}{\partial x} &= \frac{\partial a}{\partial x}(x,\frac{\xi}{\tau})-\frac{\partial a}{\partial x}(x,\phi'(x))-2\phi''(x)A(x)\phi'(x)+\mathcal{O}_\phi(\frac{1}{\tau}),\\ \frac{\partial\widetilde{q}_1}{\partial\xi} &= \frac{1}{\tau}A(x)\phi'(x),\\ \frac{\partial\widetilde{q}_1}{\partial x} &= \phi''(x)A(x)\frac{\xi}{\tau}+\langle\frac{\partial A}{\partial x}\frac{\xi}{\tau},\phi'(x)\rangle+\mathcal{O}_\phi(\frac{1}{\tau}),\end{aligned}$$

leading to

$$\begin{aligned} & \{\widetilde{q}_2,\widetilde{q}_1\}= \qquad (4.38)\\ & =\langle\frac{1}{\tau}2A(x)\frac{\xi}{\tau},\phi''(x)A(x)\frac{\xi}{\tau}\rangle+\langle\frac{1}{\tau}2A(x)\frac{\xi}{\tau},\langle\frac{\partial A(x)}{\partial x}\frac{\xi}{\tau},\phi'(x)\rangle\rangle\\ & \quad -\langle\frac{\partial a}{\partial x}(x,\frac{\xi}{\tau})-\frac{\partial a}{\partial x}(x,\phi'(x))-2\phi''(x)A(x)\phi'(x),\frac{1}{\tau}A(x)\phi'(x)\rangle+\mathcal{O}_\phi(\frac{1}{\tau^2})\\ & =\frac{2}{\tau}\Big[\langle A(x)\frac{\xi}{\tau},\phi''(x)A(x)\frac{\xi}{\tau}\rangle+\langle A(x)\phi'(x),\phi''(x)A(x)\phi'(x)\rangle+\mathcal{O}(|\phi'(x)|^3)\Big]\\ & \quad +\mathcal{O}_\phi(\frac{1}{\tau^2}).\end{aligned}$$

Now look for ϕ of the form

$$\phi(x) = e^{\beta\psi(x)}, \tag{4.39}$$

where $d\psi(x) \neq 0$. Then

$$\phi'(x) = e^{\beta\psi(x)}\beta\psi'(x), \tag{4.40}$$

$$\phi''(x) = e^{\beta\psi(x)}\beta^2\psi'(x) \otimes {}^t\psi'(x) + e^{\beta\psi(x)}\beta\psi''(x), \tag{4.41}$$

and (4.38) gives with $\xi = \tau|\phi'(x)|\eta$, $|\eta| \sim 1$:

$$\begin{aligned}\{\widetilde{q}_2, \widetilde{q}_1\} = {} & \frac{2}{\tau}e^{3\beta\psi}\Big[\beta^4|\psi'(x)|^2\langle A(x)\eta, (\psi'(x)\otimes {}^t\psi'(x))A(x)\eta\rangle \\ & +\beta^4\langle A(x)\psi'(x), (\psi'(x)\otimes {}^t\psi'(x))A(x)\psi'(x)\rangle + \mathcal{O}(\beta^3)\Big] + \mathcal{O}_{\beta,\psi}(\frac{1}{\tau^2}).\end{aligned} \tag{4.42}$$

Here $\psi'(x) \otimes {}^t\psi'(x) \geq 0$ and

$$\langle A(x)\psi'(x), (\psi'(x)\otimes {}^t\psi'(x))A(x)\psi'(x)\rangle = \langle A(x)\psi'(x), \psi'(x)\rangle^2 > 0,$$

so with $\beta > 0$ large enough, and for τ large enough depending on β, we get (4.15) with ϕ replaced by $\tau\phi$.

4.5 Estimates in a ball.

Recall the situation in Burq's theorem: $\mathcal{O} \subset\subset \mathbf{R}^n$ is open with smooth boundary, $\Omega = \mathbf{R}^n \setminus \mathcal{O}$ is connected, and $P = \sum_{|\alpha|\leq 2} a_\alpha(x;h)D^\alpha$ is a second order, elliptic, formally self-adjoint operator with smooth coefficients in $C^\infty(\Omega)$. Moreover P is equal to $-\Delta$ outside some ball $B(0,R_0)$, with $R_0 > 0$. We may assume that $\overline{\mathcal{O}} \subset B(0,R_0)$. We also recall that we take the Dirichlet realization of P, i.e. we realize P as a self-adjoint operator $L^2 \to L^2$ with domain $H^2 \cap H_0^1)(\Omega)$. We are interested in estimates for $P - \lambda^2$, when $\lambda \to +\infty$. Put $h = 1/\lambda$, and write

$$(P - \lambda^2) = \lambda^2(h^2P - 1). \tag{4.43}$$

$h^2P - 1$ is then then of the type that we considered above.

Lemma 4.4 *Let $R_2 > R_0$. Then there exists $\psi \in C^\infty(\overline{B(0,R_2)}\setminus\mathcal{O})$ such that $\psi = \mathrm{Const}$ and $\frac{\partial\psi}{\partial\nu} > 0$ on $\partial\mathcal{O}$, and $d\psi(x)$ is $\neq 0$ for all $x \in \overline{B(0,R_2)} \setminus \mathcal{O})$.*

Proof. First, we can find a smooth real function ψ_0 on $\overline{B(0,R_2)}\setminus\mathcal{O})$ such that $\psi_0=0$ and $\frac{\partial\psi_0}{\partial\nu}>0$ on $\partial\mathcal{O}$. Recall that if M is a smooth manifold and $f: M\to\mathbf{R}$ is a smooth function, then $x_0\in M$ is called a critical point, if $df(x_0)$ vanishes. Such a critical point is non-degenerate if the Hessian $f''(x_0)$ which can be defined in local coordinates or more generally as a linear map from the tangent space $T_{x_0}M$ into itself, is invertible. A Morse function is a smooth function f as above for which all critical points are non-degenerate. It is a well-known fact in differential topology that the Morse functions on M form a set which is dense in $C^\infty(M;\mathbf{R})$. Let ψ_1 be a smooth real-valued extension of ψ_0 to a neighborhood of $\overline{B(0,R_2)}\setminus\mathcal{O}$, and let ψ_2 be a Morse function on the same neighborhood which is very close to ψ_0 in the norm of $C^1(\overline{B(0,R_2)}\setminus\mathcal{O})$. Let ψ_3 be the restriction of ψ_2 to $\overline{B(0,R_2)}\setminus\mathcal{O}$. Then the critical points of ψ_3 are of finite number and away from $\partial\mathcal{O}$. We can modify ψ_3 into a new smooth function ψ_4 on $\overline{B(0,R_2)}\setminus\mathcal{O})$, which has the same critical points as ψ_3 and such that $\psi_4(x)=0$, $\frac{\partial\psi_4(x)}{\partial\nu}>0$ on $\partial\mathcal{O}$. Let $x_1,..,x_N\in\overline{B(0,R_2)}\setminus\mathcal{O})$ be the critical points of ψ_4. Let $[0,1]\ni t\mapsto x_j(t)\in B(0,R_2+1)\setminus\overline{\mathcal{O}}$ be smooth curves with $x_j(0)=x_j$, $x_j(1)\in\partial B(0,R_2+1)$. Then we can construct a smooth family of diffeomorphisms $\kappa_t:\overline{B(0,R_2)}\setminus\mathcal{O}\to\Omega_t\setminus\mathcal{O}$, where $\Omega_t\supset\overline{B(0,R_2)}$, such that $\kappa_t(x)=x$ for x near $\partial\mathcal{O}$ and such that $\kappa_t(x_j)=x_j(t)$. Then the lemma follows, if we let ψ be the restriction to $\overline{B(0,R_2)}\setminus\mathcal{O})$ of $\psi_4\circ\kappa_1^{-1}$. #

From ψ, we form $\phi=\tau e^{\beta\psi}$ as in section 4.4, and get the estimate:

$$h\int e^{2\phi/h}(|u|^2+|h\nabla u|^2)dx\le\mathcal{O}(1)\int e^{2\phi/h}|(h^2P-1)u|^2dx,\tag{4.44}$$

for all $u\in C_0^\infty(B(0,R_2)\setminus\mathcal{O})$ with $u_{|\partial\mathcal{O}}=0$. Recalling that $h=1/\lambda$, we get

$$\lambda^3\int e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx\le\mathcal{O}(1)\int e^{2\lambda\phi}|(P-\lambda^2)u|^2dx.\tag{4.45}$$

Here we want to remove the assumption that u has compact support. Take R_1 with $R_0<R_1<R_2$, and take χ with $1_{B(0,R_1)}\prec\chi\in C_0^\infty(B(0,R_2))$, and apply (4.45) to χu:

$$\begin{aligned}&\lambda^3\int_{B(0,R_1)\setminus\mathcal{O}}e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx\le\\&\mathcal{O}(1)\int_{B(0,R_2)\setminus\mathcal{O}}e^{2\lambda\phi}|(P-\lambda^2)u|^2dx+\mathcal{O}(1)\lambda^2\int_{B(0,R_1,R_2)}e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx,\end{aligned}\tag{4.46}$$

where $B(0,R_1,R_2)=\{x\in\mathbf{R}^n;\ R_1<|x|<R_2\}$ denotes the open shell of center 0 and radii R_1,R_2. To the left we estimate the exponential factor from

below by $e^{-\lambda C}$ and to the left we estime the exponenential factors from above by $e^{\lambda C}$, where $C > 0$ is a sufficiently large constant. It follows that there is a constant $C > 0$ such that

$$\int_{B(0,R_1)\setminus\mathcal{O}} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2)dx \leq \tag{4.47}$$
$$\mathcal{O}(1)e^{2\lambda C}\int_{B(0,R_2)\setminus\mathcal{O}} |(P-\lambda^2)u|^2 dx + \mathcal{O}(1)e^{2\lambda C}\int_{B(0,R_1,R_2)} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2)dx.$$

4.6 Estimate in a shell.

We work in the region $|x| > R_0$, where $h^2P - 1 = -h^2\Delta - 1$ has the semi-classical symbol $p = \xi^2 - 1$. In polar coordinates, $x = r\omega$, $\omega \in S^{n-1}$, $r > R_0$, we get

$$p = \rho^2 + \frac{(\omega^*)^2}{r^2} - 1,$$

where $(\omega^*)^2 \geq 0$ denotes the principal symbol of $-\Delta_{S^{n-1}}$. We want to make Carleman estimates with radial weight $\phi = \phi(r)$. We get

$$p_\phi = (\rho + i\phi')^2 + \frac{1}{r^2}(\omega^*)^2 - 1,$$

so

$$\begin{cases} \operatorname{Re} p_\phi = \rho^2 - (\phi')^2 + \frac{(\omega^*)^2}{r^2} - 1 \\ \operatorname{Im} p_\phi = 2\rho\phi', \end{cases} \tag{4.48}$$

$$\frac{1}{4}\{\operatorname{Re} p_\phi, \operatorname{Im} p_\phi\} = \rho^2\phi'' + (\phi''\phi' + \frac{(\omega^*)^2}{r^3})\phi'.$$

We need the last Poisson bracket to be > 0 on the joint characteristics where both expressions in (4.48) are equal to 0. We shall have $\phi' > 0$, so $p_\phi = 0$ is equivalent to

$$\rho = 0,\ \frac{(\omega^*)^2}{r^2} = 1 + (\phi')^2. \tag{4.49}$$

On this set, we need

$$\phi' > 0,\ \phi''\phi' + \frac{1 + (\phi')^2}{r} > 0. \tag{4.50}$$

With $\psi = (\phi')^2$ we get the linear system of inequalities,

$$\frac{1}{2}\psi' + \frac{1+\psi}{r} > 0,\ \psi > 0.$$

This will follow from

$$\psi' \geq -\frac{2}{r},\ \psi > 0.$$

Start by taking $\psi = 2(A - \log r)$, $A \gg 0$, for $1 \leq r \leq B < e^A$. Choosing B very close to e^A, we may have $\psi(B) > 0$ as small as we like. We can extend the definition of ψ to all r in such a way that $\psi > 0$ and

$$-\frac{2}{r} \leq \psi' \leq 0,$$

with equality to the right for $r \geq e^A$. Then $\psi = \epsilon^2$ for $r \geq e^A$ with $\epsilon > 0$ as small as we like.

In conclusion, we can find a smooth real function ϕ on $[1, +\infty[$ satisfying (4.50), and such that

$$\phi'(r) = \sqrt{2(A - \log r)},\ 1 \leq r \leq e^A - 1, \tag{4.51}$$

$$\phi' \text{ is decreasing on } [e^A - 1, +\infty[\text{ and equal to } \epsilon \text{ for } r \geq e^A. \tag{4.52}$$

Let $R_3 > R_2$. Then for $u \in C_0^\infty(B(0, R_0, R_3))$ (the open shell), we have the Carleman estimate,

$$\lambda^3 \int e^{2\lambda\phi}(|u|^2 + |\frac{1}{\lambda}\nabla u|^2)dx \leq \mathcal{O}_{A,\epsilon,R_3,R_0}(1) \int e^{2\lambda\phi}|(P - \lambda^2)u|^2 dx. \tag{4.53}$$

4.7 Estimate in a ball ∪ shell.

We shall combine (4.47), valid for functions on $B(0, R_2) \setminus \mathcal{O}$, with (4.53). We choose R_1, R_2 so that $R_0 < R_1 - 2$ and $R_2 = R_1 + 1$. Choose ϕ as in section 4.6 and such that $\phi(R_1) - \phi(R_1 - 1)$ is larger than the constant C in the exponents in (4.47). Modifying ϕ by a constant, we get $\phi \leq -\epsilon_0$, $|x| \leq R_1 - 1$, $\phi \geq C + \epsilon_0$, $|x| \geq R_1$, for some $\epsilon_0 > 0$.

Let $u \in C^\infty(B(0, R_3) \setminus \mathcal{O})$, $u_{|\partial\mathcal{O}} = 0$, with

$$(P - \lambda^2)u = v \in C_0^\infty(B(0, R_0) \setminus \mathcal{O}). \tag{4.54}$$

Then from (4.47), (4.53), we get:

$$\begin{aligned} &\int_{B(0,R_1)\setminus\mathcal{O}} (|u|^2 + \frac{1}{\lambda}\nabla u|^2)dx \leq \\ &\mathcal{O}(1)e^{2\lambda C} \int |v|^2 dx + \mathcal{O}(1)e^{2\lambda C} \int_{B(0,R_1,R_2)} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2)dx, \end{aligned} \tag{4.55}$$

$$\int_{B(0,R_1,R_3-1)} e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx \le \mathcal{O}(1)\times \tag{4.56}$$
$$\Big(\int_{B(0,R_1-2,R_1-1)} e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx + \int_{B(0,R_3-1,R_3)} e^{2\lambda\phi}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx\Big).$$

There is a $\delta > 0$ such that the last term of the right hand side of (4.55) can be estimated by $e^{-\lambda\delta}$ times the left hand side of (4.56) and the first term of the right hand side of (4.56) can be estimated by $e^{-\lambda\delta}$ times the left hand side of (4.55), when $\lambda > 0$ is large enough. So, if we sum (4.55), (4.56), we get rid of these two contributions to the the right hand sides and with

$$\phi_+(x) = \begin{cases} 0, \ |x| < R_1, \\ \phi(x), \ |x| \ge R_1, \end{cases}$$

we get

$$\int_{B(0,R_3-1)\setminus\mathcal{O}} e^{2\lambda\phi_+}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx \le \tag{4.57}$$
$$\mathcal{O}(1)e^{2\lambda C}\int |v|^2 dx + \mathcal{O}(1)\int_{B(0,R_3-1,R_3)} e^{2\lambda\phi_+}(|u|^2+|\frac{1}{\lambda}\nabla u|^2)dx,$$

for u, v as in (4.54).

4.8 Estimates for outgoing solutions.

In the previous estimate, we may assume that R_3 is as large as we like, and that $\phi(x) = \text{Const.} + \epsilon|x|$, in $B(0, \frac{R_3}{4}, \infty)$ with ϵ as small as we like. (The choice of ϵ, R_3, will only affect the "$\mathcal{O}(1)$" factors and the required largeness of λ, but not the constant C in the exponent.) Let u be as before, but assume in addition that u is outgoing: $u = R(\lambda)v$, where $v \in L^2_{\mathrm{comp}}(B(0,R_0)\setminus\mathcal{O})$. For $1_{\mathrm{supp}\, v\cup\overline{\mathcal{O}}} \prec \chi \in C_0^\infty(\mathbf{R}^n;[0,1])$, we have

$$\begin{aligned} (\frac{1}{i}[P,\chi]u|u) &= \frac{1}{i}((\chi u|Pu)) - (Pu|\chi u)) \\ &= \frac{\lambda^2}{i}((\chi u|u) - (u|\chi u)) + \frac{1}{i}((\chi u|v) - (v|\chi u)) \\ &= \frac{1}{i}((u|v) - \overline{(u|v)}) = 2\mathrm{Im}\,(u|v). \end{aligned} \tag{4.58}$$

By the Cauchy–Schwartz inequality, we have

$$(\frac{1}{i}[P,\chi]u|u) \le 2\|u\|_{B(0,R_0)}\|v\|_{B(0,R_0)}. \tag{4.59}$$

(Recall that u is outgoing, so that $u = R_0(\lambda)w$ for $|x| \geq R_0$, for some

$$w \in C_0^\infty(B(0, R_0) \setminus \overline{\mathcal{O}}).$$

Then by Proposition 2.5

$$(\frac{1}{i}[P, \chi]u|u) = C(\lambda) \int_{\partial B(0,\lambda)} |\widehat{w}(\xi)|^2 L_{\partial B(0,\lambda)}(d\xi),$$

where $C(\lambda) > 0$ and we assume that $\lambda > 0$ in order to fix the ideas. So the scalar product in (4.59) is ≥ 0.)

The semi-classical principal symbol of $\frac{1}{i}[hP, \chi]$ is $-H_p(\chi) = -2\xi \cdot \partial_x \chi(x)$, which is ≥ 0 on the outgoing trajectories (or "rays"). Here by an outgoing ray, we mean a maximal(ly extended) integral curve (necessarily a line segment) of $H_p = 2\xi \cdot \partial_x$ in

$$\{(x, \xi) \in \mathbf{R}^{2n}; \xi^2 = 1, \frac{R_3}{2} < |x| < 2R_3\}, \tag{4.60}$$

such that the backward extension (for negative times, as an integral curve of H_{ξ^2}), passes over $B(0, \frac{R_3}{4})$. By an incoming ray, we mean a maximal ray in the set (4.60) which is not an outgoing one.

If we work in the shell $\frac{R_3}{2} \leq |x| \leq 2R_3$, and consider relative sizes compared to $\|u\|_{H^1(B(0,\frac{R_3}{4},\frac{R_3}{3}))}$, then by using phase space analysis, one can show that "u is $\mathcal{O}(e^{-\lambda R_3/C})$" near the incoming trajectories (this is a consequence of the fact that u is outgoing, see [38]) and "$\mathcal{O}_\delta(e^{\delta\lambda R_3})$" for every $\delta > 0$, near the outgoing trajectories. Also u is exponentially localized to the energy surface $\xi^2 = 1$, when we restrict x to a shell as above. In view of these facts, one expects (and can indeed show) that $(\frac{1}{i}[P, \chi]u|u)$ behaves like the square of a norm of u. More precisely we have ([17]) for every $\delta > 0$ and for $\lambda \geq \lambda(\delta)$:

$$\begin{aligned} &\int_{B(0,R_3-1,R_3)} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2) dx \leq \\ &e^{-2\lambda R_3/C_0} \int_{B(0,\frac{R_3}{4},\frac{R_3}{3})} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2) dx + e^{2R_3\delta\lambda}(\frac{1}{i}[P, \chi]u|u), \end{aligned} \tag{4.61}$$

where $C_0 > 0$. Combining this with (4.57), (4.59) leads to

$$\begin{aligned} &\int_{B(0,R_3-1)\setminus\mathcal{O}} e^{2\lambda\phi_+}(|u|^2 + |\frac{1}{\lambda}\nabla u|^2) dx \leq \\ &\mathcal{O}(1)e^{2\lambda C} \int |v|^2 dx + \mathcal{O}(1)e^{2\lambda(\phi_+(R_3) - \frac{R_3}{C_0})} \int_{B(0,\frac{R_4}{4},\frac{R_3}{3})} (|u|^2 + |\frac{1}{\lambda}\nabla u|^2) dx \\ &+\mathcal{O}(1)e^{2\lambda(\phi_+(R_3)+\delta)} \|u\|_{B(0,R_0)} \|v\|_{B(0,R_0)}. \end{aligned} \tag{4.62}$$

Since $|\nabla\phi| \leq \epsilon$ for $\frac{R_3}{4} \leq |x|$, the second term to the right can be absorbed by the left hand side. So can the u-contribution to the last term, if we use the standard inequality "$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$":

$$\mathcal{O}(1)e^{2\lambda(\phi_+(R_3)+\delta)}\|u\| \leq \frac{1}{2}\|u\|^2 + \mathcal{O}(1)e^{4\lambda(\phi_+(R_3)+\delta)}\|v\|^2.$$

With a new constant C depending on R_3, we obtain:

$$\int_{B(0,R_3-1)\setminus\mathcal{O}}(|u|^2 + |\frac{1}{\lambda}\nabla u|^2)dx \leq e^{2\lambda C}\int |v|^2 dx. \tag{4.63}$$

This implies (4.4) and concludes our outline of the proof of Burq's theorem.

5 Review of non-selfadjoint spectral theory

[Missing piece: Fredholm theory via Grushin problems as in an appendix of [38].]

In this chapter we review some of the standard theory for non-selfadjoint operators, and we will essentially follow the book [31]. Let $\mathcal{H}$ be a separable complex Hilbert space. If $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$ is compact, we let $s_1(A) \geq s_2(A) \geq \dots \searrow 0$ be the eigenvalues of $(A^*A)^{1/2}$. They are called the singular values of A. We notice that $s_j(A^*) = s_j(A)$. In fact, this follows from the intertwining relations:

$$A(A^*A) = (AA^*)A,\ (A^*A)A^* = A^*(AA^*).$$

The singular values appear naturally in the *polar decomposition:* If $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$, then

$$\|Au\|^2 = (Au|Au) = (A^*Au|u) = ((A^*A)^{1/2}u|(A^*A)^{1/2}u) = \|(A^*A)^{1/2}u\|^2.$$

The operator

$$U : \mathcal{R}((A^*A)^{1/2}) \ni (A^*A)^{1/2}u \mapsto Au \in \mathcal{R}(A)$$

is isometric and bijective. It extends to a unitary operator that we also denote by U from $\overline{\mathcal{R}((A^*A)^{1/2}}$ to $\overline{\mathcal{R}(A)}$ and to a partial isometry if we put $U = 0$ on the orthogonal space $(\mathcal{R}(A^*A)^{1/2})^\perp = \mathcal{N}((A^*A)^{1/2}) = \mathcal{N}(A)$, we get the polar decomposition :

$$A = U(A^*A)^{1/2}. \tag{5.1}$$

This leads to the Schmidt decomposition of A: Let $e_1, e_2, ..$ be an orthonomral family of eigenvectors of $(A^*A)^{1/2}$ associated to the eigenvalues $s_1(A), s_2(A), ..$ that are > 0. Then

$$Au = \sum s_j(A)(u|e_j)f_j, \tag{5.2}$$

where $f_j = Ue_j$ is also an orthonormal family.

Recall the mini-max characterization of the the s_j:

$$s_j(A) = \inf_{\substack{E\subset\mathcal{H};\, E \text{ is a closed} \\ \text{subspace of } \mathcal{H} \\ \text{of codimension } \leq j-1}} \sup_{u\in E\setminus 0} \frac{((A^*A)^{1/2}u|u)}{\|u\|^2}. \tag{5.3}$$

From that we get the following characterization of the singular values which is due to Allaverdiev:

Theorem 5.1 *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be compact. Then*

$$s_{n+1}(A) = \min_{\substack{K\in\mathcal{L}(\mathcal{H},\mathcal{H}) \\ K \text{ of rank } \leq n}} \|A - K\|, \ n = 0, 1, \dots .$$

The minimum is realized by an operator K for which $s_1(K) = s_1(A)$,..,$s_n(K) = s_n(A)$, $s_{n+1}(K) = 0$, $s_1(A - K) = s_{n+1}(A)$, $s_2(A - K) = s_{n+2}(A)$,.. .

Proof. If K is of rank $\leq n$, then $\mathcal{N}(K)$ is of codimension $\leq n$ and

$$s_{n+1}(A) \leq \sup_{0\neq u\in\mathcal{N}(K)} \frac{\|Au\|}{\|u\|} = \sup_{0\neq u\in\mathcal{K}} \frac{\|(A - K)u\|}{\|u\|} \leq \|A - K\|.$$

To get the minimizing operator write the polar decomposition $A = U(A^*A)^{1/2}$ and take $K = U(A^*A)^{1/2}P_n$, where P_n is the orthogonal projection onto the space spanned by $e_1, .., e_n$. Then

$$A - K = U(A^*A)^{1/2}(1 - P_n),$$

$$(A-K)^*(A-K) = (1-P_n)(A^*A)^{1/2}U^*U(A^*A)^{1/2}(1-P_n) = (A^*A)(1-P_n),$$

and we get the statement about the singular values of $A - K$. Especially $s_{n+1}(A) = \|A - K\|$. The statement about the singular values of K can be obtained similarly. #

The following corollary is due to Ky Fan:

Corollary 5.2 *Let $A, B \in \mathcal{L}(\mathcal{H})$ be compact. Then for $n, m \geq 0$:*

$$s_{m+n-1}(A+B) \leq s_m(A) + s_n(B) \tag{5.4}$$

$$s_{m+n-1}(AB) \leq s_m(A)s_n(B). \tag{5.5}$$

Proof. Let K_A, K_B be operators of rank $\leq m-1$ and $\leq n-1$ respectively, such that

$$s_m(A) = \|A - K_A\|, \ \ s_n(B) = \|B - K_B\|.$$

Then

$$s_{m+n-1}(A+B) \leq \|A+B-(K_A+K_B)\| \leq \|A-K_A\|+\|B-K_B\| = s_m(A)+s_n(B).$$

The proof for AB is essentially the same. #

Corollary 5.3 *We have $|s_n(A) - s_n(B)| \leq \|A - B\|$.*

Proof. Let K be an operator of rank $n-1$. Then

$$s_n(A) \leq \|A - K\| = \|B - K + A - B\| \leq \|B - K\| + \|A - B\|.$$

Varying K, we get $s_n(A) \leq s_n(B) + \|A-B\|$, and we have the same inequality with A and B exchanged. #

We now discuss Weyl inequalities, and start with the following result to H. Weyl:

Theorem 5.4 *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be compact and let $\lambda_1(A), \lambda_2(A), ..$ be the non-vanishing eigenvalues of A arranged in such a way that $|\lambda_1| \geq |\lambda_2| \geq ...$ and repeated according to their multiplicity (which by definition is the rank of the spectral projection). The for every $n \geq 1$ for which $\lambda_n(A)$ is defined, we have*

$$|\lambda_1(A) \cdot .. \cdot \lambda_n(A)| \leq s_1(A) \cdot .. \cdot s_n(A). \tag{5.6}$$

(For notational reasons, we assume that there are infinitely many non-vanishing eigenvalues of A.)

Proof. For $n = 1$, (5.6) just says that $|\lambda_1(A)| \leq \|A\|$. Approaching A by a sequence of finite rank operators, we can assume that A is of finite rank and replace $\mathcal{H}$ by the finite dimensional space $\mathcal{R}(A) + (\mathcal{N}(A))^{\perp}$, that we denote by $\mathcal{H}$ from now on. [[[Haer maaste vi saega naagot om kontinueten av egenvaerden som funktioner av A.]]] Introduce the space

$$\wedge^n \mathcal{H} = \mathcal{H} \wedge ... \wedge \mathcal{H} \tag{5.7}$$

generated by n-fold exterior products of vectors in $\mathcal{H}$. $\wedge^n \mathcal{H}$ is a Hilbert space with a scalar product that satisfies

$$(u_1 \wedge .. \wedge u_n | v_1 \wedge .. \wedge v_n) = \det((u_j|v_k)),\ u_j, v_j \in \mathcal{H}. \tag{5.8}$$

Further, there is a linear operator $\wedge^n A : \wedge^n \mathcal{H} \to \wedge^n \mathcal{H}$ which is uniquely determined by the condition,

$$(\wedge^n A)(u_1 \wedge .. \wedge u_n) = Au_1 \wedge .. \wedge Au_n,\ u_j \in \mathcal{H}. \tag{5.9}$$

Using a basis of generalized eigenvectors, we see that the eigenvalues of $\wedge^n A$ are the values $\lambda_{j_1} \cdot .. \cdot \lambda_{j_n}$, with $j_\nu \neq j_\mu$, for $\nu \neq \mu$. The eigenvalue of greatest modulus is then $\lambda_1 \cdot .. \cdot \lambda_n$. On the other hand the adjoint of $\wedge^n A$ is $\wedge^n A^*$. We also have $(\wedge^n A)(\wedge^n B) = \wedge^n (AB)$. Then $(\wedge^n A)^*(\wedge^n A) = \wedge^n (A^* A)$ and this operator has the eigenvalues $s_{j_1}(A) \cdot .. \cdot s_{j_n}(A)^2$ out of which the largest one is

$$(s_1(A) \cdot .. \cdot s_n(A))^2 = \| \wedge^n A\|^2 \geq |\lambda_1| \cdot .. \cdot |\lambda_n|^2.$$

The proof is complete. #

In the same spirit we have the inequality of A. Horn:

$$\prod_1^n s_j(AB) \leq (\prod_1^n (s_j(A) s_j(B)) \tag{5.10}$$

Proof. As before it suffices to treat the case when $\mathcal{H}$ is of finite dimension. The largest eigenvalue of

$$(\wedge^n AB)^*(\wedge^n AB) = \wedge^n ((AB)^* AB)$$

is equal to $(s_1(AB) \cdot .. \cdot s_n(AB))^2$. On the other hand,

$$((\wedge^n AB)^*(\wedge^n AB)u|u) = \|(\wedge^n AB)u\|^2 = \|(\wedge^n A) \circ (\wedge^n B)u\|^2$$
$$\leq \| \wedge^n A\|^2 \| \wedge^n B\|^2 \|u\|^2 \leq (s_1(A) \cdot .. \cdot s_n(A))^n (s_1(B) \cdot .. \cdot s_n(B))^2 \|u\|^2,$$

and taking the supremum over all normalized u, we obtain the required inequality. #

We next need a convexity inequality, due to Weyl and Littlewood–Polya.

Lemma 5.5 *Let $\Phi(x)$ be a convex function on* $\mathbf{R}$*, which tends to 0, when $x \to -\infty$, with $\Phi(-\infty) := \lim_{x\to-\infty}\Phi(x) = 0$. Let $a_1 \geq .. \geq a_N$, $b_1 \geq .. \geq b_N$ be real numbers with*

$$\sum_1^k a_j \leq \sum_1^k b_j,\ 1 \leq k \leq N.$$

Then,

$$\sum_1^k \Phi(a_j) \leq \sum_1^k \Phi(b_j),\ 1 \leq k \leq N.$$

Proof. Approaching Φ by a sequence of smooth functions, we can reduce the proof to the case when $\Phi \in C^\infty$. Then $\Phi' \geq 0$,

$$\Phi'(x) \to 0,\ x \to -\infty. \tag{5.11}$$

Letting $y \to -\infty$ in the identity

$$\Phi(x) = \Phi(y) + \int_y^x \Phi'(t)dt, \tag{5.12}$$

we get

$$\Phi(x) = \int_{-\infty}^x \Phi'(t)dt.$$

From the convergence of the last integral, we conclude that $\int_y^0 \Phi'(t)dt \leq C$, implying that $|y|\Phi'(y)$, is a bounded function for $y \leq 0$, which tends to 0 when $y \to -\infty$.

Integration by parts in (5.12) gives

$$\begin{aligned}\Phi(x) &= \Phi(y) + [(t-x)\Phi'(t)]_{t=y}^x - \int_y^x (t-x)\Phi''(t)dt \\ &= \Phi(y) + (x-y)\Phi'(y) + \int_y^x (x-t)\Phi''(t)dt.\end{aligned}$$

Letting y tend to $-\infty$, we get

$$\Phi(x) = \int_{-\infty}^x (x-t)\Phi''(t)dt = \int (x-t)_+\Phi''(t)dt.$$

Hence

$$\sum_1^k \Phi(a_j) = \int (\sum_{j\ =1}^k (a_j - t)_+)\Phi''(t)dt,$$

for every t. Let $k(t) \le k$ be the largest $\widetilde{k} \le k$ with $a_{\widetilde{k}} \ge t$. Then

$$\sum_{j=1}^{k}(a_j - t)_+ = \sum_{j\ =1}^{k(t)}(a_j - t) \le \sum_{j=1}^{k(t)}(b_j - t) \le \sum_{j=1}^{k}(b_j - t)_+.$$

Hence $\sum_1^k \Phi(a_j) \le \sum_1^k \Phi(b_k)$. #

As a consequence we get the following result of Weyl :

Theorem 5.6 *Let $A : \mathcal{H} \to \mathcal{H}$ be a compact operator, and $f(x) \ge 0$ a function on $[0, \infty[$ with $f(0) = 0$ such that $f(e^t)$ is convex. Let λ_j and s_j be the eigenvalues and singular values of A, arranged with $|\lambda_1| \ge |\lambda_2| \ge ...$, $s_1 \ge s_2 \ge ....$ Then for every $k \ge 1$:*

$$\sum_1^k f(|\lambda_j|) \le \sum_1^k f(s_j). \tag{5.13}$$

Proof. We know that

$$\sum_1^k \log|\lambda_j| \le \sum_1^k \log s_j,$$

and it suffices to apply the preceding convexity lemma. #

Corollary 5.7 *For every $p > 0$, we have*

$$\sum_1^n |\lambda_j(A)|^p \le \sum_1^n s_j(A)^p.$$

For every $r > 0$, we have

$$\prod_1^n (1 + r|\lambda_j(A)|) \le \prod_1^n (1 + rs_j(A)).$$

Let A, B be compact operators. With $\Phi(t) = e^t$, $a_j = \log s_j(AB)$, $b_j = \log(s_j(A)s_j(B))$, we get from Horn's inequality (5.10) and Lemma 5.5 :

Corollary 5.8 $\sum_1^n s_j(AB) \le \sum_1^n s_j(A)s_j(B)$.

Let $C_\infty \subset \mathcal{L}(\mathcal{H})$ be the subspace of compact operators. The following Lemma is due to Ky Fan.

Lemma 5.9 *Let $A \in C_\infty$. Then for every $1 \le n \in \mathbf{N}$, we have*

$$\sum_1^n s_j(A) = \max \sum_{j=1}^n (UA\phi_j|\phi_j),$$

where the maximum is taken over the set of all unitary operators U and all orthonormal systems $\phi_1, .., \phi_n$.

We leave the proof as an exercise, or else see [31].

Corollary 5.10 *If $A, B \in \mathcal{S}_\infty$, then*

$$\sum_1^n s_j(A+B) \le \sum_1^n s_j(A) + \sum_1^n s_j(B).$$

We are no ready to discuss the Schatten–von Neumann classes.

Definition. For $1 \le p \le \infty$, we put

$$C_p = \{A \in C_\infty; \sum_1^\infty s_j(A)^p < \infty\}.$$

Theorem 5.11 *C_p is a closed two-sided ideal in $\mathcal{L}(\mathcal{H}, \mathcal{H})$ equipped with the norm*

$$\|A\|_{C_p} = \|(s_j(A))_1^\infty\|_{\ell^p}.$$

If $p_1 \le p_2$, then $C_{p_1} \subset C_{p_2}$. The space of finite rank operators is dense in C_p for every p.

We will only recall the proof of the fact that $\| \cdot \|_{C_p}$ satisfies the triangle inequality. Let $A, B \in C_p$, and put $\xi_j = s_j(A+B)$, $\eta_j = s_j(A) + s_j(B)$. According to Corollary 5.10, we have $\sum_1^n \xi_j \le \sum_1^n \eta_j$, $\forall n$ and hence,

$$\|A+B\|_{C_1} \le \|A\|_{C_1} + \|B\|_{C_1},$$

by letting n tend to ∞.

It remains to treat the case $p > 1$. ξ_j and η_j are both decreasing sequencies. It suffices to show that

$$\|\xi\|_{\ell^p} \le \|\eta\|_{\ell^p}. \tag{5.14}$$

We have

$$\|\xi\|_{\ell^p} = \sup_{\substack{\zeta \in \ell^q, \\ \|\zeta\|_{\ell^q}=1}} \langle \xi, \zeta \rangle,$$

by Hölder, where $q \in [1, +\infty[$ is the conjugate index, given by $p^{-1} + q^{-1} = 1$ and $\langle \cdot, \cdot \rangle$ denotes the real scalar product on $\ell^2(\{1, 2, ..\})$. We also know that the supremum is attained by a $\zeta = \zeta^0$ of the form $\zeta_j^0 = (\text{Const.} > 0)\xi_j^{p/q}$ and in particular ζ_j^0 is a decreasing sequence. We use partial summation with $\Xi_j = \sum_1^j \xi_k$, $\Xi_0 = 0$:

$$\begin{aligned}\langle \xi, \zeta^0 \rangle^{(n)} : &= \sum_1^n \xi_j \zeta_j^0 = \sum_{j=1}^n (\Xi_j - \Xi_{j-1})\zeta_j^0 \\ &= \sum_{j=1}^n \Xi_j \zeta_j^0 - \sum_{j=1}^{n-1} \Xi \zeta_{j+1}^0 = \Xi_n \zeta_n^0 + \sum_{j=1}^{n-1} \Xi_j (\zeta_j^0 - \zeta_{j+1}^0) \\ &= (\sum_1^n \xi_k)\zeta_n^0 + \sum_{j=1}^{n-1} \sum_{k=1}^{j} \xi_k) \underbrace{(\zeta_j^0 - \zeta_{j+1}^0)}_{\geq 0}\end{aligned}$$

The last expression is $\leq$ the same expression with ξ replaced by η and running the same calculation backwards the latter expression is equal to $\langle \eta, \zeta^0 \rangle^{(n)}$ Hence $\langle \xi, \zeta^0 \rangle^{(n)} \leq \langle \eta, \zeta^0 \rangle \leq \|\eta\|_{\ell^p}$. Letting n tend to infinity, we get (5.14), and this completes the proof of the triangle-inequality for the C_p-norms. #

We notice that $\|A\| = s_1(A) \leq \|A\|_{C_p}$. The space C_1 is the space of nuclear or trace-class operators, and C_2 is the space of Hilbert-Schmidt operators. We have the following Hölder type result :

Theorem 5.12 *Let $p, q \in [1, \infty]$ be conjugate indices; $p^{-1} + q^{-1} = 1$. If $A \in C_p$, $B \in C_q$, then $AB \in C_1$ and $\|AB\|_{C_1} \leq \|A\|_{C_p} \|B\|_{C_q}$.*

Proof. We know that

$$\sum_1^n s_j(AB) \leq \sum_1^n s_j(A)s_j(B)$$

and letting n tend to ∞, we get from the usual Hölder inequuality:

$$\sum_1^\infty s_j(AB) \leq \sum_1^\infty s_j(A)s_j(B) \leq \|s_\cdot(A)\|_{\ell^p} \|s_\cdot(B)\|_{\ell^q} = \|A\|_{C_q} \|B\|_{C_q}.$$

#

We next discuss *the trace* of a nuclear operator. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is of finite rank, we choose a finite dimensional subspace $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{R}(A) \subset \mathcal{H}'$,

$(\mathcal{H}')^\perp \subset \mathcal{N}(A)$ We then define the trace of A, $\mathrm{tr}\, A$ as the trace $\mathrm{tr}\, A_{|_{\mathcal{H}'}}$ of the restriction of A to $\mathcal{H}'$. We check that this does not depend on the choice of $\mathcal{H}'$ and that $\mathrm{tr}\, A$ is the sum of the finitely many non-vanishing eigenvalues of A (each counted with its algebraic multiplicity). We see that

$$A \mapsto \mathrm{tr}\, A \tag{5.15}$$

is a linear functional on the space of finite rank operators. Moreover,

$$|\mathrm{tr}\, A| \le \sum |\lambda_j(A)| \le \|A\|_{C_1}. \tag{5.16}$$

We can then extend (5.15) to a continuous linear functional on C_1 and we still have

$$|\mathrm{tr}\, A| \le \|A\|_{C_1}. \tag{5.17}$$

In the case of finite rank operators, we also have

$$\mathrm{tr}\, AB = \mathrm{tr}\, BA. \tag{5.18}$$

Let now $A \in C_p$, $B \in C_q$, where $p, q \in [1, \infty]$ are conjugate indices and choose A_ν, B_ν, $\nu = 1, 2, ..$ of finite rank, so that $\|A - A_\nu\|_{C_p} \to 0$, $\|B - B_\nu\|_{C_q} \to 0$. Then

$$\begin{aligned} \|AB - A_\nu B_\nu\|_{C_1} &= \|(A - A_\nu)B + A_\nu(B - B_\nu)\|_{C_1} \\ &\le \|A - A_\nu\|_{C_p}\|B\|_{C_q} + \|A_\nu\|_{C_p}\|B - B_\nu\|_{C_q} \to 0,\ \nu \to \infty. \end{aligned}$$

Using this also for BA and the cyclictiy of the trace (5.18) for finite rank operators, we obtain it also in the case $A \in C_p$, $B \in C_q$, where $p, q \in [1, \infty]$ are conjugate indices.

Remark. There is a simple way of extending most of the theory to the case of operators $A : \mathcal{H}_1 \to \mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are two different Hilbert spaces. Consider namely the corresponding operator

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{5.19}$$

and say that A belongs to C_p if the operator in (5.19) does. We leave these extensions as an exercise for the reader, and notice simply that the cyclicity of the trace still holds in this setting, namely if $A : \mathcal{H}_1 \to \mathcal{H}_2$, and $B : \mathcal{H}_2 \to \mathcal{H}_1$ belong to C_p and C_q respectively, where p and q are conjugate indices.

We next discuss determinants of trace-class perturbations of the identity operator. Let first $A \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be of finite rank and chose a finite dimensional Hilbert space as above. Then we define

$$\det(1 - A) = \det((1 - A)_{|\mathcal{H}'}) = \prod_j (1 - \lambda_j(A)), \tag{5.20}$$

where $\lambda_j(A)$ denote the non-vanishing eigenvalues, repeated according to their multiplicity. We remark that

$$|\det(1 - A)| \leq \prod_j (1 + |\lambda_j(A)|) \leq \prod_j (1 + s_j(A)) \leq e^{\sum_j s_j(A)},$$

where the the second inequlity follows from Corollary 5.7 . We want to extend the definition to the case when $A \in C_1$. Let first I be a compact interval and let $I \ni t \mapsto A_t$ be a $C^1 family$ of finite rank rank operators, with $\mathcal{R}(A_t) \subset \mathcal{H}'$, $\mathcal{N}(A_t) \subset (\mathcal{H}')^\perp$ for some finite dimensional subspace $\mathcal{H}'$ which is independent of t. We first assume that $1 - A_t$ is invertible for all $t \in I$, or in other words that $1 - \lambda_j(A_t) \neq 0$ for all t and j. Then $\det(1 - A_t) \neq 0$ and by a classical formula,

$$\frac{\partial}{\partial t} \log\det(1 - A_t) = -\mathrm{tr}\,((1 - A_t)^{-1} \frac{\partial}{\partial t}(A_t)) = -\mathrm{tr}\,((\frac{\partial}{\partial t} A_t)(1 - A_t)^{-1}).$$

Hence,

$$\Big|\frac{\frac{\partial}{\partial t}\det(1 - A_t)}{\det(1 - A_t)}\Big| = \Big|\frac{\partial}{\partial t}\log\det(1 - A_t)\Big| \leq \|(1 - A_t)^{-1}\| \|\frac{\partial}{\partial t} A_t\|_{C_1}.$$

In particular, if $I = [0, 1]$, $A_t = tA_1 + (1 - t)A_0$, we get

$$|\log\det(1 - A_1) - \log\det(1 - A_0)| \leq \sup_{0 \leq t \leq 1} \|(1 - (tA_1 + (1 - t)A_0))^{-1}\| \|A_1 - A_0\|_{C_1}. \tag{5.21}$$

Now let $A \in C_1$. If $1 - A_1$ is not invertible, we put $\det(1 - A) = 0$. Assume then that $1 - A$ is invertible. Let A_ν be a sequence of finite rank operators which converges to A in C_1. For ν large enough, we have $\|(1 - A_\nu)^{-1}\| \leq C_0$ for some fixed constant C_0 and more generally $\|(1 - (tA_\nu + (1 - t)A_0))^{-1}\| \leq C_0$. Then

$$|\log\det(1 - A_\nu) - \log\det(1 - A_\mu)| \leq C_0 \|A_\nu - A_\mu\|_{C_1}$$

and consequently $\lim_{\nu\to\infty}(\log\det(1-A_\nu)-\log\det(1-A_\mu))$ exists. We then put

$$\det(1-A)=\det(1-A_\mu)\exp\lim_{\nu\to\infty}(\log\det(1-A_\nu)-\log\det(1-A_\mu)). \quad (5.22)$$

Notice that

$$|\det(1-A)|\le\prod(1+s_j(A))\le e^{\|A\|^{C_1}}. \quad (5.23)$$

Using approximation by finite rank operators, we also see that

$$\det((1-A)(1-B))=\det(1-A)\det(1-B). \quad (5.24)$$

By the same argument, we can extend (5.21) to general trace clacc operators for which $1-(tA_1+(1-t)A_0)$ is invertible.

We now add a complex variable $z\in\mathbf{C}$ and consider the function $\det(1-zA)$. If A is of finite rank, then this is an entire function of z. If $A\in C_1$, let $A_\nu\to A$ be a sequence of finite rank operators. Then $|\det(1-zA_\nu)|\le e^{|z|\|A_\nu\|_{C_1}}$. If $z\lambda_j(A)\ne 1$, $\forall j$, then $\det(1-zA_\nu)\to\det(1-zA)$ with locally uniform convergence in $\mathbf{C}\setminus\cup_j\{1/\lambda_j\}$, and it follows that $\det(1-zA)$ is a holomorphic function on this set, which verifies

$$|\det(1-zA)|\le e^{\|A\|_{C_1}|z|}. \quad (5.25)$$

It follows that $\det(1-zA_\nu)$ converges to an entire function $f(z)$ locally uniformly on $\mathbf{C}$. If $z=1/\lambda_j(A)$ where $\lambda_j(A)$ is of multiplicity m, then exacly m eigenvlues of A_ν will converge to $\lambda_j(A)$ while the others will stay away from a neighborhood of this point (when ν is large enough). Considering the argument variation (Rouché), we conclude that $f(z)$ vanishes to the order m at $1/\lambda_j(A)$ and in particular we have $f(z)=\det(1-zA)$ also at that point. In conclusion, we have

Proposition 5.13 *Let $A\in C_1$. Then $D_A(z):=\det(1-zA)$ is an entire function whose zeros counted with multiplicity coincide with the values $1/\lambda_1(A),1/\lambda_2(A),..$ counted with the multiplicities of $\lambda_1(1),\lambda_2(A),...$*

Observe that

$$D_A(0)=1 \quad (5.26)$$

Also observe that $D_A(z)$ is of subexponental growth in the sense that for every $\epsilon>0$ there exists a constant $C_\epsilon>0$ such that

$$|D_A(z)|\le C_\epsilon e^{\epsilon|z|}. \quad (5.27)$$

In fact, by a limiting argument, we have

$$|D_A(z)| \le \prod_1^\infty (1+|z|s_j(A)) \le \prod_1^N (1+|z|s_j(A)) e^{\sum_{N+1}^\infty s_j(A))|z|}.$$

Here the prefactor is of polynomial growth for every fixed N and for a given $\epsilon > 0$, we can always choose $\mathbf{N} > 0$ so large that the exponent is $\le \epsilon|z|$.

The next observation is that $W(z) = \prod_1^\infty (1 - z\lambda_j(A))$ is also an entire function of subexponential growth. This follows by the same argument, if we recall that $\sum_1^\infty |\lambda_j(A)|$ is convergent. We now use a special case of a theorem of Hadamard (see [2]): Since $D_A(z)$ and $W(z)$ have the same zeros (counted with multiplicity), we have

$$D_A(z) = W(z)e^{g(z)},$$

where $g(z)$ is an entire function. It is also clear that we can choose g with $g(0) = 0$. The function

$$\operatorname{Re} g(z) := \log|D_A(z)| - \sum_1^\infty \log|1 - \lambda z| \tag{5.28}$$

is harmonic. Let $R \ge 2$. For $|z| \le R/2$, we have

$$\operatorname{Re} g(z) = \int_{|w|=R} P_R(z,w) \operatorname{Re} g(w) \ell(dw), \tag{5.29}$$

where P_R is the Poisson kernel for the disc of radius R and ℓ denotes the length element on the boundary of this disc. By a scaling argument, it is easy to see that

$$\frac{1}{CR} \le P_R(z,w) \le \frac{C}{R},\ |z| \le \frac{R}{2},\ |w| = R, \tag{5.30}$$

where $C > 0$ is independent of R. Using the subexponential growth of $D_A(z)$, we get

$$\int_{|w|=R} P_R(z,w) \log|D_A(w)| \ell(dw) \le \frac{C}{R}\epsilon R R \le C\epsilon R, \tag{5.31}$$

for $R \ge R_\epsilon$ large enough. On the other hand,

$$|\int_{|w|=R} P_R(z,w) \sum_1^\infty \log|1 - \lambda_j w| \ell(dw)| \le \sum_1^\infty \frac{1}{R} \int_{|w|=R} |\log|1 - \lambda_j w|| \ell(dw) \tag{5.32}$$

If $|\lambda_j|R \leq \frac{1}{2}$, we write $|\log|1-\lambda_j w|| \leq C|\lambda_j|R$ and the sum over the corresponding js in (5.32) can be bounded by

$$\sum_{|\lambda_j|\leq \frac{1}{2R}} \frac{C}{R} 2\pi |\lambda_j| R R = 2\pi C R \sum_{|\lambda_j|\leq \frac{1}{2R}} |\lambda_j| = o(R),\ R\to\infty. \tag{5.33}$$

If $\frac{1}{2} \leq |\lambda_j|R \leq T$, where $T \gg 1$ is independent of R, then with $a = |\lambda_j|R \in [\frac{1}{2}, T]$:

$$\frac{1}{R}\int_{|w|=R} |\log|1-\lambda_j w||\ell(dw) = \int_{|\zeta|=1} |\log|1-a\zeta||\ell(d\zeta).$$

With $\zeta = e^{i\theta}$, we get

$$\begin{aligned}&|1-a\zeta|^2 = (1-a\cos\theta)^2 + a^2(\sin\theta)^2\\ &= 1+a^2-2a\cos\theta = a(\frac{1}{a}+a-2\cos\theta) \geq 2a(1-\cos\theta).\end{aligned}$$

Consequently,

$$\frac{1}{R}\int_{|w|=R} |\log|1-\lambda_j w||\ell(dw) \leq C_T.$$

Let us estimate the number of λ_j in this case:

$$\sum_{\frac{1}{2}\leq|\lambda_j|R\leq T} 1 \leq 2R \sum_{\frac{1}{2R}\leq|\lambda_j|\frac{T}{R}} |\lambda_j| = o_T(R).$$

Hence

$$\sum_{\frac{1}{2}\leq|\lambda_j|R\leq T} \frac{1}{R}\int_{|w|=R} |\log|1-\lambda_j w||\ell(dw) = o_T(R),\ R\to\infty. \tag{5.34}$$

It remains to consider the case $|\lambda_j|R \geq T$. Here $\log|1-\lambda_j R| \sim \log(|\lambda_j|R)$. Hence, with constants C and C_δ that are independent of T:

$$\begin{aligned}&\sum_{|\lambda_j|R\geq T} \frac{1}{R}\int_{|w|=R} |\log|1-\lambda_j w||\ell(dw) \leq C \sum_{|\lambda_j|R\geq T} \log(|\lambda_j|R)\\ &\leq C_\delta \sum_{|\lambda_j|\geq\frac{T}{R}} |\lambda_j|^\delta R^\delta = C_\delta R^\delta \sum_{|\lambda_j|\geq\frac{T}{R}} |\lambda_j|^\delta.\end{aligned}$$

Put $\delta = 1/p$, $1 < p < \infty$, and let q be the conjugate index. Then, if N denotes the number of λ_j with $|\lambda_j| \geq T/R$, we get from Hölder's inequlity:

$$\sum_{|\lambda_j|\geq \frac{T}{R}} |\lambda_j|^{\frac{1}{p}} \leq N^{\frac{1}{q}} (\sum_{|\lambda_j|\geq \frac{T}{R}} |\lambda_j|)^{\frac{1}{p}}. \tag{5.35}$$

Here $NT/R \leq \sum |\lambda_j| \leq C$, so $N \leq CR/T$ and the expression (5.35) is bounded by $CR^{1/q}/T^{1/q}$. Hence

$$\sum_{|\lambda_j|R\geq T} \frac{1}{R} \int_{|w|=R} |\log|1-\lambda_j w||\ell(dw) \leq CR^{\frac{1}{p}+\frac{1}{q}}/T^{\frac{1}{q}} = \frac{CR}{T^{\frac{1}{q}}}. \tag{5.36}$$

Combining the three cases, we find

$$|\int_{|w|=R} P_R(z,w)(\sum_1^\infty \log|1-\lambda_j w|)\ell(dw)| \leq o_T(R) + \frac{CR}{T^{\frac{1}{q}}} = o(R),\ R\to\infty \tag{5.37}$$

Combining this with (5.28), (5.29) and (5.31), we get

$$\operatorname{Re} g(z) \leq o(R), \text{ on } |z| \leq \frac{R}{2}.$$

Now we can apply Harnack's inequality to the function $\operatorname{Re} g - o(R)$, which is ≤ 0 on the disc $|z| \leq R/2$ and $\geq -o(R)$ at 0 and conclude that

$$\operatorname{Re} g \geq -o(R) \text{ on the disc } |z| \leq \frac{R}{4}.$$

Since g is harmonic, it follows from the last two estimates that $\operatorname{Re} g = 0$. Hence g is constant and since we have chosen g with $g(0) = 0$, we get $g(z) = 0$ $\forall z \in \mathbf{C}$. We have then showed

Theorem 5.14 *Let $A : \mathcal{H} \to \mathcal{H}$ be a trace class operator with the non-vanishing eigenvalues $\lambda_1(A), \lambda_2(A), ..,$ $0 \leq N \leq \infty$ repeated according to multiplicity. Then $D_A(z) = \det(1 - zA)$ satisfies*

$$D_A(z) = \prod_{j=1}^N (1-\lambda_j z),\ z \in \mathbf{C}. \tag{5.38}$$

From this we get the important Lidskii's theorem as a corollary :

Corollary 5.15 *If $A \in C_1$ we have*

$$\operatorname{tr} A = \sum_1^N \lambda_j(A), \tag{5.39}$$

where $\lambda_j(A)$ are the non-vanishing eigenvalues as in Theorem 5.14.

Proof. We know the result when A is of finite rank. In this case, we also know that

$$\frac{D_A'}{D_A} = \frac{\partial}{\partial z} \log \det(1 - zA) = -\operatorname{tr}((1 - zA)^{-1}A),$$

away from the zeros of $z \mapsto \det(1 - zA)$. In particular,

$$\frac{D_A'(0)}{D_A(0)} = -\operatorname{tr} A, \tag{5.40}$$

when A is of finite rank.

When $A \in C_1$, let A_ν be a sequence of finite rank operators converging to A in the C_1 norm. Then $D_{A_\nu}(z) \to D_A(z)$, $D_{A_\nu}'(z) \to D_A'(z)$, when $\nu \to \infty$, uniformly for z in a neighborhood of 0. Since $D_A(z) \neq 0$ in such a neighborhood, we have also

$$\frac{D_{A_\nu}'(0)}{D_{A_\nu}(0)} \to \frac{D_A'(0)}{D_A(0)}.$$

By (5.40) we know that the right hand side of the last relation is equal to $-\operatorname{tr} A_\nu$ and we also know that this quantity converges to $-\operatorname{tr} A$. Consequently (5.40) remains valid for general trace class operators. In view of Theorem 5.14, we know on the other hand that

$$\frac{D_A'(0)}{D_A(0)} = -\sum_1^N \lambda_j(A),$$

and the Corollary follows. #

The last proof also shows that

$$\frac{D_A'(z)}{D_A(z)} = \frac{\partial}{\partial z} \log D_A(z) = -\operatorname{tr}(1 - zA)^{-1}A = -\sum_1^N \frac{\lambda_j(A)}{1 - z\lambda_j(A)}, \tag{5.41}$$

for all z with $1 - z\lambda_j(A) \neq 0$, $\forall j$.

We now turn to some other questions, that will be of use.

Proposition 5.16 *Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ be complex separable Hilbert spaces. Let $\Omega \subset \mathbf{C}^n$ be an open set and let $\Omega \ni z \mapsto K(z) \in C_1(\mathcal{H}_1)$ be a holomorphic function. Then* $\det(1 - K(z))$ *is a holomorphic function on Ω and if $z_0 \in \Omega$ is a zero of order $m \geq 1$ of this function, and $A(z) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $B(z) \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ depend holomorphically on $z \in \Omega$, then*

$$\operatorname{rank}\left(\int_\gamma A(z)(1 - K(z))^{-1} B(z) dz\right) \leq m, \tag{5.42}$$

if γ is the positively oriented boundary of a sufficiently small circle centered at z_0.

Proof. We know that 1 is an eigenvalue of $K(z_0)$ of a certain multiplicity N_0, defined to be the rank of the spectral projection

$$\Pi(z_0) = \frac{1}{2\pi i} \int_\alpha (\lambda - K(z_0))^{-1} d\lambda,$$

where α is the positively oriented boundary of a small disc centered at $\lambda = 1$. For z close to z_0, we put

$$\Pi(z) = \frac{1}{2\pi i} \int_\alpha (\lambda - K(z))^{-1} d\lambda,$$

and we notice that this is the sum of the spectral projections corresponding to the N_0 eigenvalues $\lambda_{j_0}(z), \lambda_{j_0+1}(z), .., \lambda_{j_0+N_0-1}(z)$ (repeated according to multiplicity) that are close to $\lambda_{j_0}(z_0)$. m is then also the order of vanishing of $(1 - \lambda_{j_0}(z)) \cdot .. \cdot (1 - \lambda_{j_0+N_0-1}(z))$ (if we note that $\det(1 - K(z)) = \det(1 - K(z)\Pi(z)) \det(1 - K(z)(1 - \Pi(z)))$). The range of $\Pi(z)$ is of constant dimension N_0 and we can find a basis $e_1(z), .., e_{N_0}(z)$ of this space which depends holomorphically on z (possibly after restricting z to a new even smaller neighborhood of z_0.

Define $R_+ : \mathcal{H}_1 \to \mathbf{C}^{N_0}$ by $R_+(z)u(j) = a_j(u, z)$, where $\Pi(z)u = \sum_1^{N_0} a_j(u, z) e_j$. Define $R_-(z) : \mathbf{C}^{N_0} \to \mathcal{H}_1$ by $R_-(z)u_- = \sum_1^{N_0} u_-(j) e_j(z)$. Then

$$\begin{pmatrix} 1 - K(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{H}_1 \times \mathbf{C}^{N_0} \to \mathcal{H}_1 \times \mathbf{C}^{N_0}$$

is bijective with inverse

$$\begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E^{-+}(z) \end{pmatrix}$$

where E_{-+} is the matrix of the restriction of $K(z)-1$ to $\mathcal{R}(\Pi(z))$ with respect to the basis $e_1(z),..,e_{N_0}(z)$. Hence $\det E_{-+}(z) = \prod_{\nu=0}^{N_0-1}(1-\lambda_{j_0}(z))...(1-\lambda_{j_0+N_0-1}(z))$ has the same order of vanishing at $z=z_0$ as $\det(1-K(z))$. It then suffices to apply Lemma [[... som skall ingaa i en foersta del av detta kapitel daer vi behandlar Fredholm teori med hjaelp av Grushin problem]].

We end this chapter by recalling Jensen's formula and the standard application to getting bounds on the number of zeros of holomorphic functions. Let $f(z)$ be a holomorphic function on the open disc $D(0,R)$ with a continuous extension to the corresponding closed disc. Assume that $f(0)\neq 0$ and $f(z)\neq 0$ for $|z|=R$.

Assume first that $f(z)$ has no zeros at all. Then $\log|f(z)| = \mathrm{Re}\,\log f(z)$ is a harmonic function in the open disc which is continuous up to the boundary, and the mean value property of harmonic functions tells us that

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta.$$

We now allow f to vanish and let $z_1,..,z_N$ be the zeros repeated according to their multiplicity. Since f is not allowed to vanish at 0 or at the boundary of the disc of radius R, we have $0<|z_j|<R$. Then

$$F(z) := f(z)\prod_{j=1}^{N}\frac{R^2-\overline{z_j}z}{R(z-z_j)}$$

is holomorphic in the open disc, continuous up to the boundary and has no zeros in the closed disc of radius R. Moreover $|F(z)|=|f(z)|$ when $|z|=R$, so according to the preceding paragraph, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta.$$

Expanding the left hand side, we get Jensen's formula :

$$\log|f(0)| + \sum_1^N \log\frac{R}{|z_j|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta. \tag{5.43}$$

A standard application of this formula is to notice that if $N(R/2)$ is the number of zeros z_j of f with $|z_j|\leq R/2$, then we get

$$N(\frac{R}{2})\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta - \log|f(0)|. \tag{5.44}$$

6 Global upper bounds on the number of resonances

In this chapter we shall show that in standard situations and for odd dimensions ≥ 3, the number of resonances in a disc $D(0, r)$ (of center 0 and radius r) is $\mathcal{O}(r^n)$, $r \to \infty$. It will be convenient to formulate the result in the black box setting of section 2.3 of chapter 2, and after that we will mention some of the history of this result. The methods come to a large extent from the theory of non-selfadjoint operators, and can therfore be viewed as classical.

Let $P : \mathcal{H} \to \mathcal{H}$ be as in chapter 2, section 2.3, so that $\mathcal{H}$ is given by the orthogonal decomposition (2.30). Introduce the torus $M = (\mathbf{R}/2R\mathbf{Z})^n$ with $R > R_0$ and identify $\overline{B(0, R_0)}$ with its image under the natural projection $\mathbf{R}^n \to M$. Put $\mathcal{H}^\sharp \to \mathcal{H}^\sharp$. The domain $\mathcal{D}^\sharp = \mathcal{D}(P^\sharp)$ is by definition

$$\mathcal{D}^\sharp = \{\chi u_1 + (1 - \chi)u_2;\ u_1 \in \mathcal{D},\ u_2 \in H^2(M)\}. \tag{6.1}$$

Here $\chi \in C_0^\infty$ has its support in a small neighborhood of $\overline{B(0, R_0)}$ and is equal to one near that set. We view χ both as an element of $C_0^\infty(\mathbf{R}^n)$ and as an element of $C_0^\infty(M)$. It is easy to see that the definition (6.1) does not depend on the choice of χ. For $\mathcal{D}(P^\sharp) \ni u = \chi u_1 + (1 - \chi)u_2$, we put

$$P^\sharp u = P(\chi u_1) + (-\Delta)((1 - \chi)u_2), \tag{6.2}$$

and again, this does not depend on the representation of u.

Lemma 6.1 *$P^\sharp$ is self-adjoint and has purely discrete spectrum.*

Proof. It is easy to see that $P^\sharp$ is symmetric and we leave the details to the reader. Let $u \in \mathcal{D}((P^\sharp)^*)$, $(P^\sharp)^* u = v$, $v \in \mathcal{H}^\sharp$. We claim that $u \in \mathcal{D}(P^\sharp)$: First we use that

$$(u| - \Delta\phi) = (v|\phi),\ \forall \phi \in C_0^\infty(M \setminus \overline{B(0, R_0)}),$$

so that $-\Delta u = v$ in $M \setminus \overline{B(0, R_0)}$ and hence

$$u_{|M\setminus\overline{B(0,R_0)}} \in H^2_{\mathrm{loc}}(M \setminus \overline{B(0, R_0)}).$$

After subtracting an exterior part from u, we may assume that $\chi u = u$, with χ as above. Now we can view u, v as elements of $\mathcal{H}$, and from the relation $(P^\sharp)^* u = v$, it follows that

$$(u|P\phi) = (v|\phi),$$

first for all $\phi \in \mathcal{D}$ with support in a neighborhodd of $\mathrm{supp}\,\chi$, then for all $\phi \in \mathcal{D}$. Since P is self-adjoint, we get $u \in \mathcal{D}$ and the claim follows.

Let $v_j \in \mathcal{H}^\sharp$, $j = 1, 2, ...$ be a bounded sequence and consider $u_j = (P^\sharp + i)^{-1} v_j \in \mathcal{D}^\sharp$. With χ as before, we first see that $(1-\chi)(P^\sharp + i)^{-1} v_j = (1-\chi) u_j$ is bounded in $H^2(M)$ and hence has a convergent subsequence in $\mathcal{H}^\sharp$. As for χu_j, we compute $(P + i)\chi u_j = [-\Delta, \chi] u_j + \chi v_j =: w_j$, so w_j is bounded in $\mathcal{H}$. Now write $\chi u_j = (P + i)^{-1} w_j = (P + i)^{-1} \widetilde{\chi} w_j$, where $\chi \prec \widetilde{\chi} \in C_0^\infty(\mathbf{R}^n)$. Since $(P + i)^{-1}\widetilde{\chi}$ is compact, χu_j has a convergent subsequence in $\mathcal{H}$ and in $\mathcal{H}^\sharp$. It follows that $(P^\sharp + i)^{-1} : \mathcal{H}^\sharp \to \mathcal{H}^\sharp$ is compact. Hence $P^\sharp$ has purely discrete spectrum. #

Let $N(P^\sharp, I)$ be the number of eigenvalues of $P^\sharp$ in the interval I. We assume

$$N(P^\sharp, [-\lambda, \lambda]) = \mathcal{O}(1)\Phi(\lambda), \ \lambda \geq 1, \tag{6.3}$$

where $\Phi : [1, \infty[\to [1, \infty[$ is continuous strictly increasing, with $\Phi(1) = 1$. We also assume:

$$\Phi(t) \geq t^{\frac{n}{2}}, \tag{6.4}$$

$$\begin{gathered} \forall C \geq 1, \ \exists \widetilde{C}(C) \geq 1, \text{ such that} \\ \Phi(Ct) \leq \widetilde{C}(C)\Phi(t), \ \phi(C\lambda) \leq \widetilde{C}(C)\phi(\lambda), \ \lambda, t \geq 1, \end{gathered} \tag{6.5}$$

where $\phi(\lambda) = \Phi^{-1}(\lambda)$.

Let $\lambda_1, \lambda_2, \lambda_3, ...$ be the eigenvalues of $P^\sharp$ repeated according to multiplicity and arranged so that $|\lambda_1| \leq |\lambda_2| \leq ....$ The eigenvalues of the compact normal operator $(i - P^\sharp)^{-1}$ are then $(i - \lambda_j)^{-1}$, and the corresponding singular values are given by

$$s_j((i - P^\sharp)^{-1}) = \frac{1}{|i - \lambda_j|} = \frac{1}{\langle \lambda_j \rangle}. \tag{6.6}$$

It is easy to see that the property (6.3) is equivalent to

$$s_j((i - P^\sharp)^{-1}) \leq \frac{\mathcal{O}(1)}{\phi(j)}. \tag{6.7}$$

For the same operator P, let $\widetilde{M}$ be a second torus and $\widetilde{P}$ the corresponding operator analogous to $P^\sharp$. We identify M and $\widetilde{M}$ by means of a diffeomorphism which acts like the identity near $\overline{B(0, R_0)}$. In this way, we can view both P and $P^\sharp$ as operators on M (i.e. as operators $\mathcal{H}^\sharp \to \mathcal{H}^\sharp$.

(Notice however, that $\widetilde{P}$ will not be self-adjoint with respect to the scalar product of $\mathcal{H}^\sharp$ but for another one which gives an equivalent norm.) We have the resolvent identity:

$$(i-\widetilde{P})^{-1}=(i-P^\sharp)^{-1}+(i-\widetilde{P})^{-1}(\widetilde{P}-P^\sharp)(i-P^\sharp)^{-1}. \tag{6.8}$$

Now recall some facts from Chapter 5 about singular values. If $K:\mathcal{H}\to\mathcal{H}$ is a compact operator, then the jth singular value (i.e. the jth eigenvalue of $\sqrt{K^*K}$ counted with multiplicities in decreasing order) obeys Ky Fan's identity

$$s_j(K)=\inf_{\substack{R\in\mathcal{L}(\mathcal{H},\mathcal{H}),\\ \mathrm{rank}\,(R)\le j-1}}\|K-R\|. \tag{6.9}$$

From this we deduce the wellknown inequalities:

$$s_{j+k-1}(K+L)\le s_j(K)+s_k(L), \tag{6.10}$$

$$s_{j+k-1}(KL)\le s_j(K)s_k(L), \tag{6.11}$$

for $j,k\ge 1$. Applying this to (6.8), we see that $\widetilde{P}$ also satisfies (6.7). This shows that the assumption (6.3) does not depend on the choice of M.

Later on we will also need the following information which follows from (6.9): If K is a compact operator, then for every $j\ge 1$, there exists a bounded operator R of rank $\le j-1$, such that

$$\begin{aligned} s_\nu(K) &= s_\nu(R),\ 1\le\nu\le j-1,\\ s_j(K) &= \|K-R\|=s_1(K-R),\\ s_\nu(K) &= s_{\nu-j+1}(K-R),\ \nu\ge j. \end{aligned} \tag{6.12}$$

The same argument shows that

$$s_j((P-z)^{-1}\chi),\ s_j(\chi(P-z)^{-1})\ \le\frac{C}{\phi(j)}, \tag{6.13}$$

for $\chi\in C_0^\infty(\mathbf{R}^n)$, $1_{B(0,R_0)}\prec\chi$ and for z in any fixed comapct subset of the resolvent set. This estimate is the starting point of the proof of the following result:

Theorem 6.2 *Let $n\ge 3$ be odd. Under the assumptions above, the number $N(r)$ of resonances in the open disc $D(0,r)$ satisfies*

$$N(r)\le C\Phi(r^2),\ r\ge 1. \tag{6.14}$$

Before starting the proof we give some examples and make some historical remarks. Let g be a smooth Riemannian metric on $\mathbf{R}^n$ which coincides with the standard Euclidean metric on $\mathbf{R}^n$ outside a bounded set, let $\mathcal{O} \subset\subset \mathbf{R}^n$ be open with smooth boundary, let $V \in L^\infty_{\mathrm{comp}}(\mathbf{R}^n \setminus \mathcal{O})$, and let P be the Dirichlet or Neumann realization in $L^2(\mathbf{R}^n \setminus \mathcal{O})$ of the Schrödinger operator $-\Delta_g + V(x)$, where Δ_g denotes the Laplace-Beltrami operator for the metric g. Then we can take $\Phi(r) = r^{n/2}$ and we get $N(r) \leq Cr^n$. In the case, $V = 0$ and $\Delta_g = \Delta$ everywhere, this was proved by Melrose [54]. Zworski [101] obtained it for $-\Delta + V$ without any obstacle, and Vodev [92] obtained it for $-\Delta_g$ on $L^2(\mathbf{R}^n)$. Theorem 6.2 as stated above is essentially due to Sjöstrand–Zworski [80], but in the proof below we shall mainly follow Vodev [93], who extended the result to certain non-selfadjoint operators and who also obtained sharp results in the even-dimensional case [95, 96]. In the context of hyperbolic manifolds, analogous results have been obtained by Guillopé–Zworski [34], Froese–Hislop [25].

Proof. Let

$$1_{B(0,R_0)} \prec \chi_0 \prec \chi_1 \prec \chi_2,\ \chi_j \in C_0^\infty(\mathbf{R}^n)$$

as in Chapter 2. Let $\mu \in \mathbf{C}$, $\arg \mu = \frac{\pi}{4}$, $|\mu| = r \gg 1$. Then we know that the resonances λ are contained in the set of λ, such that $(1 + K(\lambda, \mu)\chi) : \mathcal{H} \to \mathcal{H}$ is not invertible, where $\chi_2 \prec \chi \in C_0^\infty(\mathbf{R}^n)$. Here K is the same as in Chapter 2, (2.39), (2.40), (2.41):

$$K(\lambda, \mu) = [\Delta, \chi_0]R_0(\lambda)(1-\chi_1) - [\Delta, \chi_2]R(\mu)\chi_1 + (\mu^2 - \lambda^2)\chi_2 R(\mu)\chi_1. \quad (6.15)$$

We will restrict the attention to $\{\lambda \in \mathbf{C};\ |\lambda - \mu| < 3r\}$. Since $K(\lambda, \mu)\chi$ is not necessarily of trace class, we shall first decompose it into one term with small norm, and one term which is of trace class. This decomposition only concerns the last term in (6.15).

Write

$$R(\mu)\chi_1 = (P - i|\mu|^2)^{-1}(P - i)(P - i)^{-1}\chi,$$

and conclude that

$$s_j(\chi_2 R(\mu)\chi_1) \leq \min\left(\frac{1}{r^2}, \frac{\mathcal{O}(1)}{\phi(j)}\right),\ j = 1, 2, ... \quad (6.16)$$

For $j(r) = \mathcal{O}(1)\Phi(r^2)$, we can decompose :

$$\chi_2 R(\mu)\chi_1 = A_\mu + B_\mu, \quad (6.17)$$

where

$$\|A\| \leq \frac{1}{100r^2}, \tag{6.18}$$

$$s_j(B) = \begin{cases} s_j(\chi_2 R(\mu)\chi_1), \ 1 \leq j \leq j(r), \\ 0, \ j \geq j(r)+1. \end{cases} \tag{6.19}$$

We have already seen in Chapter 2, that if $|\mu| = r$ is large enough, then

$$\|[\Delta, \chi_2]R(\mu)\chi_1\| \leq \frac{1}{4}. \tag{6.20}$$

Write

$$\begin{aligned} &1 + K(\lambda,\mu) = \\ &1 + D + (\mu^2 - \lambda^2)B + [\Delta, \chi_0]R_0(\lambda)(1-\chi_1)\chi, \end{aligned} \tag{6.21}$$

where $D = (\mu^2 - \lambda^2)A - [\Delta, \chi_2]R(\mu)\chi_1$ and $\|D\| \leq 1/2$,

$$\begin{aligned} 1 + K(\lambda,\mu)\chi &= (1+D)(1+\widetilde{K}), \\ \widetilde{K} &= (\mu^2-\lambda^2)(1+D)^{-1}B + (1+D)^{-1}[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi. \end{aligned} \tag{6.22}$$

We shall take the determinant of $1+\widetilde{K}$, and in order to estimate this determinant, we shall first estimate the singular values of the last term in the expression for $\widetilde{K}$.

For $\operatorname{Im}\lambda \geq 0$, $\alpha > 0$, we have

$$s_j([\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi) \leq C_\alpha \langle\lambda\rangle^\alpha j^{-\alpha/n}, \tag{6.23}$$

and for $\operatorname{Im}\lambda < 0$:

$$s_j([\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi) \leq \begin{cases} e^{C\langle\lambda\rangle}, \ \forall j \\ j^{-\frac{n}{n-1}} + C_\alpha\langle\lambda\rangle^\alpha j^{\frac{-\alpha}{n}}, \ j \geq C\langle\lambda\rangle^{n-1}. \end{cases} \tag{6.24}$$

We postpone the proof of these two estimates and finish the proof of the theorem. Let

$$h(\lambda,\mu) = \det(1+\widetilde{K})$$

For $\lambda = \mu$, we get

$$h(\mu,\mu) = \det(1 + (1+D)^{-1}[\Delta,\chi_0]R_0(\mu)(1-\chi_1)\chi).$$

In this case we can use the off diagonal exponential decay of the distribution kernel of $R_0(\mu)$, to see that the distribution kernel $k(x,y)$ of $[\Delta,\chi_0]R_0(\mu)(1-\chi_1)\chi$ satsifies:

$$\sum_{|\alpha|\le 2n+1}\|\partial^\alpha_{x,y}k(x,y)\|_{L^1(\mathbf{R}^{2n})}\le \mathcal{O}(1)e^{-r/C_0}, \tag{6.25}$$

for some $C_0>0$. On the other hand it is well known that the trace class norm of $[\Delta,\chi_0]R_0(\mu)(1-\chi_1)\chi$ is bounded by a constant times the left hand side of (6.25), so

$$\|(1+D)^{-1}[\Delta,\chi_0]R_0(\mu)(1-\chi_1)\chi\|_{\mathrm{tr}}\le \mathcal{O}(1)e^{-r/C_0}. \tag{6.26}$$

Use also the Weyl inequality

$$|\det(1+\widetilde{K})|\le \prod_1^\infty(1+s_j(\widetilde{K}))\le e^{\|\widetilde{K}\|_{\mathrm{tr}}}, \tag{6.27}$$

to conclude that

$$\frac{1}{C}\le |h(\mu,\mu)|\le C, \tag{6.28}$$

for $r=|\mu|$ sufficiently large.

We shall next find upper bounds for $|h(\lambda,\mu)|$. It follows from (6.23), (6.24), that (6.24) holds also in the upper half plane. We may assume that $\|(1+D)^{-1}\|\le 2$ and get

$$|h(\lambda,\mu)|\le \prod_{j=1}^\infty(1+2s_j((\mu^2-\lambda^2)B+[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi)).$$

Use that for $j=2k-1$: $s_j(A_1+A_2)\le s_k(A_1)+s_k(A_2)$ by Corollary 5.2:

$$\begin{aligned}|h(\lambda,\mu)| &\le \prod_{k=1}^\infty(1+2s_{2k-1}((\mu^2-\lambda^2)B)+[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi))^2\\ &\le \prod_{k=1}^\infty(1+2s_k((\mu^2-\lambda^2)B))^2\prod_{k=1}^\infty(1+2s_k([\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi))^2\\ &= F_1F_2,\end{aligned}$$

where the last equation defines the two factors F_j in the natural way. Clearly, $F_1\le \exp(\mathcal{O}(1)r^2\sum_1^\infty s_k(B))$. Hence by (6.16), (6.19):

$$\sum_1^\infty s_k(B)\le \sum_1^{j(r)}\min(\frac{1}{r^2},\frac{\mathcal{O}(1)}{\phi(k)})\le \sum_1^{\mathcal{O}(1)\Phi(r^2)}\frac{1}{r^2}\le \mathcal{O}(1)\frac{\Phi(r^2)}{r^2}.$$

Hence

$$F_1 \le \exp(C_1 \Phi(r^2)). \tag{6.29}$$

Using (6.24):

$$F_2 \le (\prod_{1\le k\le C\langle\lambda\rangle^{n-1}} e^{2C\langle\lambda\rangle}) \times$$

$$\exp(4 \sum_{k>C\langle\lambda\rangle^{n-1}} j^{-\frac{n}{n-1}} + C_{\alpha_1} \sum_{C\langle\lambda\rangle^{n-1}\le j\le\langle\lambda\rangle^n} \langle\lambda\rangle^{\alpha_1} j^{-\frac{\alpha_1}{n}} + C_{\alpha_2} \sum_{j>\langle\lambda\rangle^n} \langle\lambda\rangle^{\alpha_2} j^{-\frac{\alpha_2}{n}}),$$

where we are free to choose $\alpha_1, \alpha_2 > 0$. Take $\alpha_1 < n < \alpha_2$. Then evaluating the three sums in the last exponent, we get

$$F_2 \le \exp C\langle\lambda\rangle^n. \tag{6.30}$$

Consequently, for $|\lambda - \mu| \le 3r$, $\arg \mu = \frac{\pi}{4}$, $|\mu| = r$:

$$|h(\lambda, \mu)| \le e^{C(r^n + \Phi(r^2))} \le e^{\widetilde{C}\Phi(r^2)}. \tag{6.31}$$

by Jensen's formula for the disc $D(\mu, 3r)$, we see that the number of zeros of $\lambda \to h(\lambda, \mu)$ in $D(\mu, 2r)$ is $\le \mathcal{O}(1)\Phi(r^2)$. Now write (cf (2.46))

$$\begin{aligned} R(\lambda) &= Q(\lambda,\mu)(1 + K(\lambda,\mu)\chi)^{-1}(1 - K(\lambda,\mu)(1-\chi)) \\ &= Q(\lambda,\mu)(1 + \widetilde{K}(\lambda,\mu))^{-1}(1+D)^{-1}(1 - K(\lambda,\mu)(1-\chi)) \\ &= A(\lambda,\mu)(1 + \widetilde{K}(\lambda,\mu))^{-1}B(\lambda,\mu), \end{aligned}$$

where A, B are holomorphic in λ. After multiplying to the left and to the right by functions in C_0^∞, we can apply the propositions 5.16, 2.3, and conclude that the number of resonances in $D(\mu, 2r)$ is bounded by the number of zeros of $h(\cdot, \mu)$ in that set and hence by $\mathcal{O}(1)\Phi(r^2)$. The theorem follows. #

It remains to prove (6.23), (6.24). We start with the first of these estimates. For $|\lambda| \le 1$, we use that $[\Delta, \chi_0]R_0(\lambda)(1-\chi_1)\chi$ has a distribution kernel in C_0^∞. If Ω is a large ball, we can consider our operator on Ω and write

$$[\Delta, \chi_0]R_0(\lambda)(1-\chi_1)\chi = (1-\Delta_\Omega)^{-N}(1-\Delta_\Omega)^N[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi,$$

where Δ_Ω denotes the Dirichlet Laplacian in Ω. Here

$$(1-\Delta_\Omega)^N[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi$$

is uniformly bounded and the jth singular value of $(1-\Delta_\Omega)^{-N}$ is $\sim j^{-\frac{2}{n}N}$ by the known (Weyl) asymptotics for the eigenvalues of $-\Delta_\Omega$.

For $|\lambda|\geq 1$, we follow the same idea, but we have to be more precise. Let us first show that if $\chi_1,\chi_2\in C_0^\infty$, then

$$\chi_1 R_0(\lambda)\chi_2=\mathcal{O}_s(\frac{1}{|\lambda|}):\ H_s\to H_s,\ \ s\in\mathbf{R},\ \mathrm{Im}\,\lambda\geq 0. \tag{6.32}$$

Indeed, on the Fourier transform side, the kernel of $\chi_1R_0(\lambda)\chi_2$ becomes

$$K(\xi,\eta)=\int\widehat{\chi}_1(\xi-\zeta)\frac{1}{\zeta^2-\lambda^2}\widehat{\chi}_2(\zeta-\eta)\frac{d\zeta}{(2\pi)^n},\ \ \xi,\eta\in\mathbf{R}^n.$$

If $|\zeta^2-\lambda^2|\geq\frac{|\lambda|}{C}$, $\zeta\in\mathbf{R}^n$, then $|K(\xi,\eta)|\leq C_N\frac{\langle\xi-\eta\rangle^{-N}}{|\lambda|}$ and $\chi_1R(\lambda)\chi_2=\mathcal{O}_s(\frac{1}{|\lambda|}):H^s\to H^s$, $\forall s$.

If $|\zeta^2-\lambda^2|$ is smaller than $\frac{|\lambda|}{C}$, then $|\zeta|\sim|\lambda|$, and hence $|\nabla_\zeta(\zeta^2-\lambda^2)|\sim|\lambda|$. We then replace $\mathbf{R}^n_\zeta$ by the deformed contour:

$$\Gamma:\mathbf{R}^n\ni\zeta\mapsto\zeta\pm i\epsilon\chi(\frac{|\zeta|^2-|\lambda|^2}{|\lambda|^2})\frac{\zeta}{|\zeta|}=\zeta+i\mathcal{O}(\epsilon)=:\widetilde{\zeta},$$

for a suitable χ, and along Γ, we have

$$\widetilde{\zeta}^2=\zeta^2\pm 2i\epsilon\chi(\ldots)\frac{\zeta^2}{|\zeta|}-\epsilon^2\chi^2(\ldots),$$

and we get $|\widetilde{\zeta}^2-\lambda^2|\geq|\lambda|/C$. Using Paley–Wiener, we obtain (6.32).

Next we show that if $\chi_1,\chi_2\in C_0^\infty(\mathbf{R}^n)$, $\mathrm{supp}\,\chi_1\cap\mathrm{supp}\,\chi_2=\emptyset$, then

$$\chi_1R_0(\lambda)\chi_2=\mathcal{O}_{s,m}(\langle\lambda\rangle^{m-1}):H^s\to H^{s+m},\ \text{for } m\geq 0. \tag{6.33}$$

In fact, if $m=2k$, we notice that

$$(-\Delta+1)R_0(\lambda)=(1+\lambda^2)R_0(\lambda)+1,$$

and hence outside $\mathrm{supp}\,\chi_2$:

$$(-\Delta+1)R_0(\lambda)\chi_2=(1+\lambda^2)R_0(\lambda)\chi_2.$$

By iteration, we get toutside $\mathrm{supp}\,\chi_2$:

$$(-\Delta+1)^kR_0(\lambda)\chi_2=(1+\lambda^2)^kR_0(\lambda)\chi_2,$$

so

$$\chi_1(-\Delta+1)^k R_0(\lambda)\chi_2 = (1+\lambda^2)^k \chi_1 R_0(\lambda)\chi_2.$$

This gives (6.33) for $m = 2k$, and by interpolation, we get it for general $m \geq 0$.

Proof of (6.23). Let Ω be as before, so that the jth eigenvalue of $(-\Delta_\Omega + 1)^{1/2}$ is $\sim j^{1/n}$. Then as before,

$$[\Delta, \chi_0]R_0(\lambda)(1-\chi_1)\chi = (1-\Delta_\Omega)^{-\frac{\alpha}{2}} \underbrace{(1-\Delta_\Omega)^{\frac{\alpha}{2}}[\Delta,\chi_0]R_0(\lambda)(1-\chi_1)\chi}_{=\mathcal{O}(\langle\lambda\rangle^\alpha) \text{ in operator norm}}.$$

It suffices to use $s_j((1-\Delta_\Omega)^{-\frac{\alpha}{2}}) \sim j^{-\frac{\alpha}{n}}$ #

Proof of (6.24). Thanks to (2.5), (6.23) it suffices to show that

$$s_j(q_1 T(\lambda) q_0) \leq \begin{cases} e^{C\langle\lambda\rangle}, \ \forall j, \\ j^{-\frac{n}{n-1}}, \ j \geq C\langle\lambda\rangle^{n-1}, \end{cases} \tag{6.34}$$

where q_1, q_0 are differential operators with C_0^∞ coefficients of order ≤ 1 and 0 respectively and where $T(\lambda)$ is given by (2.6). Clearly, $\|T(\lambda)\| \leq e^{C\langle\lambda\rangle}$ so the first estimate in (6.34) is obvious.

We observe that $T(\lambda) = \lambda^{n-2}E(\overline{\lambda})^* E(\lambda)$, where

$$E(\lambda)u(\omega) = \int e^{-i\lambda\omega\cdot y}\chi(y)u(y)dy. \tag{6.35}$$

Write

$$T(\lambda) = \lambda^{n-2}E(\overline{\lambda})^*(1-\Delta_{S^{n-1}})^{-k}(1-\Delta_{S^{n-1}})^k E(\lambda). \tag{6.36}$$

Here

$$s_j((1-\Delta_{S^{n-1}})^{-k}) \leq (\frac{j}{C})^{-\frac{2k}{n-1}}, \tag{6.37}$$

$$\|(1-\Delta_{S^{n-1}})^k E(\lambda) q_0\| \leq e^{2|\lambda|}(Ck)^{2k}, \tag{6.38}$$

where the last estimate follows from the Cauchy inequalities. We deduce that

$$\begin{aligned} s_j(q_1 T(\lambda) q_0) &\leq Ce^{3|\lambda|}\Big(\frac{C}{j}\Big)^{\frac{2k}{n-1}}(Ck)^{2k} \\ &\leq Ce^{3|\lambda|}\Big(\frac{\widetilde{C}k}{j^{1/(n-1)}}\Big)^{2k}. \end{aligned} \tag{6.39}$$

Choose k so that

$$\frac{\widetilde{C}k}{j^{1/(n-1)}}=\frac{1}{e},$$

i.e. so that

$$k=\frac{j^{1/(n-1)}}{\widetilde{C}e}.$$

Actually we have to modify this choice a little bit since k should be an integer.) Then (6.39) gives

$$s_j(T(\lambda))\le C^{3|\lambda|}e^{-2\frac{j^{\frac{1}{n-1}}}{\widetilde{C}e}}. \tag{6.40}$$

For $j^{\frac{1}{n-1}}\gg|\lambda|$, we get

$$s_j(T(\lambda))\le Ce^{-\frac{j^{\frac{1}{n-1}}}{\widetilde{C}e}},$$

which is even better than (6.34). This completes the proof of (6.24). #

7 Resonances for long range perturbations of $-h^2\Delta$, analytic distorsions

7.1 Operators of black box type

Our operators will depend on a Planck's constant $h\in]0,h_0]$, $h_0>0$. Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition

$$\mathcal{H}=\mathcal{H}_{R_0}\oplus L^2(\mathbf{R}^n\setminus B(0,R_0)),\ B(0,R_0)=\{x\in\mathbf{R}^n;\ |x|<R_0\},\ R_0>0. \tag{7.1}$$

The orthogonal projections will be denoted by $u\mapsto u_{|B(0,R_0)}$, $u\mapsto u_{|\mathbf{R}^n\setminus B(0,R_0)}$. Sometimes we let the characteristic functions $1_{B(0,R_0)}$, $1_{\mathbf{R}^n\setminus B(0,R_0)}$ indicate the same projections.

Consider an unbounded self-adjoint operator depending on h:

$$P:\mathcal{H}\to\mathcal{H} \tag{7.2}$$

with domain $\mathcal{D}=\mathcal{D}(P)$. Recall that $\mathcal{D}$ is a Hilbert space with norm $\|u\|_{\mathcal{D}}=\|(P+i)u\|_{\mathcal{H}}$. Let $H^s(\mathbf{R}^n)$ be the standard Sobolev space, but equipped with the h-dependent norm $\|u\|_{H^s} = \|\langle hD\rangle^s u\|_{L^2}$. Assume that for every $1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbf{R}^n)$ independent of h,

$$\begin{aligned} &\text{the multiplication operators } 1-\chi :\ L^2(\mathbf{R}^n) \to \mathcal{H},\ \mathcal{H} \to L^2(\mathbf{R}^n) \\ &\text{restrict to bounded operators } H^2(\mathbf{R}^n) \to \mathcal{D},\ \mathcal{D} \to H^2(\mathbf{R}^n), \\ &\text{with norms bounded uniformly in } h. \end{aligned} \tag{7.3}$$

Assume

$$\chi(P+i)^{-1} \text{ is compact,} \tag{7.4}$$

for every χ as in (7.3). This assumption will be strengthened later. Further assume that

$$(Pu)_{|\mathbf{R}^n\setminus B(0,R_0)} = Q(u_{|\mathbf{R}^n\setminus B(0,R_0)}),\ u \in \mathcal{D}, \tag{7.5}$$

where

$$\begin{aligned} &Qu = \sum_{|\alpha|\le 2} a_\alpha(x;h)(hD_x)^\alpha u, \\ &a_\alpha(x;h) = a_\alpha(x) \text{ is independent of } h \text{ for } |\alpha| = 2, \\ &\text{and } a_\alpha \text{ is bounded in } C_b^\infty(\mathbf{R}^n) \text{ when } h \text{ varies .} \end{aligned} \tag{7.6}$$

We further assume that Q is formally self-adjoint on $\mathbf{R}^n$ with

$$\textstyle\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \frac{1}{C}|\xi|^2, \tag{7.7}$$

$$\textstyle\sum_{|\alpha|\le 2} a_\alpha(x;h)\xi^\alpha \to \xi^2,\ |x| \to \infty,\ \text{uniformly w.r.t. } h. \tag{7.8}$$

Example 1. Let $\mathcal{O} \subset\subset \mathbf{R}^n$ be an open set with smooth boundary and let P be the Dirichlet or Neumann realization of $-\Delta$ in $L^2(\mathbf{R}^n \setminus \mathcal{O})$. Then (cf. Chapter 2), $h^2 P$ satisfies the assumptions above, with $\mathcal{H} = L^2(\mathbf{R}^n \setminus \mathcal{O})$. (Take R_0 large enough so that the ball of radius R_0 contains $\mathcal{O}$.)

Example 2. $P = -h^2\Delta + V(x)$, where $V \in C_b^\infty(\mathbf{R}^n)$ with $V(x) \to 0$, $x \to \infty$. $\mathcal{H} = L^2(\mathbf{R}^n)$.

Let $R > R_0$, $T = (\mathbf{R}/\widetilde{R}\mathbf{Z})^n$, $\widetilde{R} > 2R$. We can view $B(0,R)$ as a subset of T. Put

$$\mathcal{H}^\sharp := \mathcal{H}_{R_0} \oplus L^2(T \setminus B(0,R_0)), \tag{7.9}$$

with orthogonal decomposition. We define a self-adjoint operator

$$P^\sharp : \mathcal{H}^\sharp \to \mathcal{H}^\sharp$$

as follows: The domain $\mathcal{D}^\sharp = \mathcal{D}(P^\sharp)$ is by definition

$$\{u \in \mathcal{H}^\sharp; \chi u \in \mathcal{D},\ (1-\chi)u \in H^2(T)\}$$

where $1_{B(0,R_0)} \prec \chi \in C_0^\infty(B(0,R);[0,1])$. (This does not depend on the choice of χ.) Let

$$Q^\sharp u = \sum_{|\alpha|\le 2} a_\alpha^\sharp(x;h)(hD)^\alpha$$

be a formally selfadjoint differential operator on T satisfying (7.6), (7.7) and with $a_\alpha^\sharp(x;h) = a_\alpha(x;h)$ for $|x| < R$. With χ as in the definition of $\mathcal{D}^\sharp$, we put

$$P^\sharp u = P\chi u + Q^\sharp(1-\chi)u,\ \ u \in \mathcal{D}^\sharp. \tag{7.10}$$

We have

Proposition 7.1 *$P^\sharp$ is self-adjoint with discrete spectrum.*

Proof. It is easy to see that $P^\sharp$ is symmetric. Let $u \in \mathcal{D}((P^\sharp)^*)$, and write $(P^\sharp)^* u = v$, with $v \in \mathcal{H}^\sharp$. Then

$$(u|Q^\sharp\phi) = (v|\phi),$$

for all $u \in C_0^\infty(T \setminus \overline{B(0,R_0)})$, so $Q^\sharp u = v$ in $T \setminus B(0,R_0)$, in the sense of distributions. From the ellipticity of the operator in this region, it follows that

$$u_{|_{T\setminus\overline{B(0,R_0)}}} \in H^2_{\mathrm{loc}}(T \setminus \overline{B(0,R_0)}).$$

After subtracting an exterior part from u, of class H^2 and hence belonging to $\mathcal{D}(P^\sharp)$, we may assume that $\chi u = u$, with χ as in the definition of $\mathcal{D}(P^\sharp)$.

We now have

$$(u|P\phi) = (v|\phi), \tag{7.11}$$

for all $\phi \in \mathcal{D}$ with support in $B(0,2R)$. (Here we may view u, v as elements of $\mathcal{H}$ since their supports are contained in $B(0,R)$.) Because of the support properties of u, v, we see that (7.11) extends to general $\phi \in \mathcal{D}$ and the selfadjointness of P implies that $u \in \mathcal{D}$, and hence $u \in D^\sharp$ by the support property of u. This finishes the proof of the self-adjointness of $P^\sharp$.

To prove that $P^\sharp$ has discrete spectrum, we shall use the assumption (7.4). Let $v_j \in \mathcal{H}^\sharp$, $j = 1, 2, ..$ be a bounded sequence and put $u_j = (P^\sharp + i)^{-1} v_j \in \mathcal{D}^\sharp$. With χ as above, we notice that $(1-\chi)(P^\sharp + i)^{-1} v_j = (1-\chi)u_j$ is a bounded sequence in $H^2(T)$ and hence has a convergent subsequence in $\mathcal{H}^\sharp$. As for χu_j, we compute

$$(P+i)\chi u_j = \underbrace{[Q,\chi]u_j}_{\text{bounded in }\mathcal{H}} + \underbrace{\chi v_j}_{\text{bounded in }\mathcal{H}} =: w_j,$$

so $\chi u_j = (P+i)^{-1}w_j = (P+i)^{-1}\widetilde{\chi}w_j$, if $\chi \prec \widetilde{\chi} \in C_0^\infty$. Now we notice that by the resolvent identity, (7.4) implies the compactness of $\chi(P-z)^{-1}$ for every $z \in \mathbf{C}\setminus\sigma(P)$. Here we can replace χ by $\widetilde{\chi}$ and the compactness of $\widetilde{\chi}(P-i)^{-1}$ implies that of $(P+i)^{-1}\widetilde{\chi}$ by passing to adjoints. Hence χu_j also has a convergent subsequence. It follows that $(P^\sharp+i)^{-1}$ is compact or equivalently, that $P^\sharp$ has a purely discrete spectrum. #

Later on we shall need assumptions on the density of eigenvalues of $P^\sharp$. Let $N(P^\sharp, I)$ denote the number of eigenvalues of $P^\sharp$ in the interval I. We assume

$$N(P^\sharp,[-\lambda,\lambda]) = \mathcal{O}(1)\Phi(\frac{\lambda}{h^2}),\ \lambda \geq 1. \tag{7.12}$$

Here $\Phi : [1,\infty[\to [1,\infty[$ is a continuous strictly increasing function with $\Phi(t) \geq t^{n/2}$ such that:

$$\text{For every } C\geq 1, \text{ there exists } \widetilde{C}(C)>0, \text{ such that } \Phi(Ct)\leq \widetilde{C}(C)\Phi(t). \tag{7.13}$$

Let us next verify the invariance of the last assumption. Consider two self-adjoint reference operators $P^\sharp$, $P^\flat$ for the same operator P, on two different torii which both contain $\overline{B(0,R_0)}$. Then $P^\sharp$ and $P^\flat$ coincide near $\overline{B(0,R_0)}$. We make all the general assumptions above for both operators, except (7.12), and we shall see in the end that if $P^\sharp$ satisfies (7.12), then so does $P^\flat$. After applying a diffeomorphism which is equal to the identity near $\overline{B(0,R_0)}$, we may assume that $T^\sharp = T^\flat =: T$ so that $P^\sharp$ and $P^\flat$ live in the same Hilbert space $\mathcal{H}^\sharp = \mathcal{H}^\flat$ and have the same domain $\mathcal{D}^\sharp = \mathcal{D}^\flat$. Since we cannot take the diffeomorphism with Jacobian 1 in general, the two Hilbert spaces will carry different though equivalent norms uniformly with respect to h.

Let z vary in a bounded subset Ω of $\mathbf{C}\setminus\mathbf{R}$. Let $Q^\sharp$, $Q^\flat$ be elliptic differential operators on T, self-adjoint for the respctive inner products, which coincide with $P^\sharp$ and $P^\flat$ respectively in $T\setminus\overline{B(0,R_0)}$. Let $\psi_0 \prec \psi_1 \prec \psi_2$ belong to $C^\infty(T;[0,1])$ with ψ_0 equal to 1 near the closure of $B(0,R_0)$ and with $P^\sharp = P^\flat$ near $\operatorname{supp}\psi_2$. As an approximation for $(z-P^\flat)^{-1}$, we try

$$E^\flat = \psi_2(z-P^\sharp)^{-1}\psi_1 + (1-\psi_0)(z-Q^\flat)^{-1}(1-\psi_1). \tag{7.14}$$

Then

$$(z-P^\flat)E^\flat(z)=1-[Q^\sharp,\psi_2](z-P^\sharp)^{-1}\psi_1+[Q^\sharp,\psi_0](z-Q^\flat)^{-1}(1-\psi_1). \quad (7.15)$$

For the last two terms we use telescopic formulas: Let $\theta_0\prec\theta_1\prec..\prec\theta_N$ be in $C^\infty(T)$ with $\operatorname{supp}\theta_N\cap\overline{B(0,R_0)}=\emptyset$. Iterating the identity

$$(z-P^\sharp)^{-1}\theta_j=\theta_{j+1}(z-Q^\sharp)^{-1}\theta_j+(z-P^\sharp)^{-1}[Q^\sharp,\theta_{j+1}](z-Q^\sharp)^{-1}\theta_j,$$

and similarly with $P^\sharp$ replaced by $Q^\sharp$, and where θ_j may be replaced by $[Q^\sharp,\theta_j]$, we get

$$\begin{aligned}&(z-P^\sharp)^{-1}\theta_0=\\ &\textstyle\sum_{j=1}^N\theta_j(z-Q^\sharp)^{-1}[Q^\sharp,\theta_{j-1}](z-Q^\sharp)^{-1}[Q^\sharp,\theta_{j-2}]..[Q^\sharp,\theta_1](z-Q^\sharp)^{-1}\theta_0\\ &\quad+(z-P^\sharp)^{-1}[Q^\sharp,\theta_N](z-Q^\sharp)^{-1}[Q^\sharp,\theta_{N-1}]..[Q^\sharp,\theta_1](z-Q^\sharp)^{-1}\theta_0.\end{aligned} \quad (7.16)$$

The adjoint formula is:

$$\begin{aligned}&\theta_0(z-P^\sharp)^{-1}=\\ &\textstyle\sum_{j=1}^N\theta_0(z-Q^\sharp)^{-1}[\theta_1,Q^\sharp]..[\theta_{j-2},Q^\sharp](z-Q^\sharp)^{-1}[\theta_{j-1},Q^\sharp](z-Q^\sharp)^{-1}\theta_j\\ &\quad+\theta_0(z-Q^\sharp)^{-1}[\theta_1,Q^\sharp]..[\theta_{N-1},Q^\sharp](z-Q^\sharp)^{-1}[\theta_N,Q^\sharp](z-P^\sharp)^{-1}.\end{aligned} \quad (7.17)$$

For the second term to the right in (7.15), take $\theta_0,..,\theta_N$ as above with $\theta_0=1$ near $\operatorname{supp}[\mathrm{Q}^\sharp,\psi_2]$ and with $\operatorname{supp}\theta_{\mathrm{N}}\cap\operatorname{supp}\psi_1=\emptyset$. Then (7.17) gives

$$\begin{aligned}&[Q^\sharp,\psi_2](z-P^\sharp)^{-1}\psi_1=\\ &\pm[Q^\sharp,\psi_2]\theta_0(z-Q^\sharp)^{-1}[Q^\sharp,\theta_1]..[Q^\sharp,\theta_{N-1}](z-Q^\sharp)^{-1}[Q^\sharp,\theta_N](z-P^\sharp)^{-1}\psi_1.\end{aligned} \quad (7.18)$$

Here $\|[Q^\sharp,\theta_N](z-P^\sharp)^{-1}\psi_1\|_{\mathcal{L}(\mathcal{H},L^2)}\le C/|\operatorname{Im}\mathrm{z}|$, where we used the assumption (7.3) which implies the corresponding fact for $P^\sharp$. On the other hand,

$$\begin{aligned}&[Q^\sharp,\psi_2]\theta_0(z-Q^\sharp)^{-1}[Q^\sharp,\theta_1]...[Q^\sharp,\theta_{N-1}](z-Q^\sharp)^{-1}\\ &\qquad=\mathcal{O}_N(1)\tfrac{h^N}{|\operatorname{Im}\mathrm{z}|^{\mathrm{N}}}:L^2\to H^N,\end{aligned}$$

where we equip H^N with the h-dependent norm $\|\langle hD\rangle^N u\|_{L^2}$. Choosing $N > n$ it follows that the last operator is of trace class as an operator in L^2 with trace class norm bounded by $\mathcal{O}_N(1)h^{N-n}/|\mathrm{Im}\, z|^N$. We have then showed that $[Q^\sharp, \psi_2](z - P^\sharp)^{-1}\psi_1$ is trace negligible (cf.[80]) in the sense that for every $N \in \mathbf{N}$ there exists $M(N) \geq 0$, such that

$$\|[Q^\sharp, \psi_2](z - P^\sharp)^{-1}\psi_1\|_{\mathrm{tr}} = \mathcal{O}_N(1)\frac{h^N}{|\mathrm{Im}\, z|^{M(N)}}.$$

Here $\|\cdot\|_{\mathrm{tr}} = \|\cdot\|_{C_1}$ denotes the trace class norm. Similarly we get the same fact for
$[Q^\sharp, \psi_0](z - Q^\flat)^{-1}(1 - \psi_1)$.

From (7.14), (7.15), we get

$$(z - P^\flat)^{-1} = \psi_2(z - P^\sharp)^{-1}\psi_1 + (1 - \psi_0)(z - Q^\flat)^{-1}(1 - \psi_1) + R^\flat(z), \quad (7.19)$$

where $R^\flat$ is trace negligible. Actually we can replace $(z-P^\sharp)^{-1}$ by $(z-P^\flat)^{-1}$, changing only the trace negligible part. Similarly,

$$(z - P^\sharp)^{-1} = \psi_2(z - P^\sharp)^{-1}\psi_1 + (1 - \psi_0)(z - Q^\sharp)^{-1}(1 - \psi_1) + R^\sharp(z), \quad (7.20)$$

where $R^\sharp(z)$ is trace negligible. If $f \in C_0^\infty(\mathbf{R})$, we have the simple Cauchy formula ([22]), much exploited since the work of [39]:

$$f(P^\sharp) = -\frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{z}}(z - P^\sharp)^{-1}L(dz), \quad (7.21)$$

valid more generally for $P^\sharp$ replaced by any self-adjoint operator. Here $\widetilde{f} \in C_0^\infty(\mathbf{C})$ is an almost analytic extension of f, i.e. with $\partial\widetilde{f}/\partial\overline{z}$ vanishing to infinite order on $\mathbf{R}$, and $L(dz)$ is the Lebesgue measure on $\mathbf{C}$. It follows from this and (7.19), (7.20), that

$$f(P^\sharp) - f(P^\flat) = (1 - \psi_0)(f(Q^\sharp) - f(Q^\flat))(1 - \psi_1) + K,$$

where $\|K\|_{\mathrm{tr}} = \mathcal{O}(h^\infty)$. On the other hand, we know that $\|f(Q^\sharp)\|_{\mathrm{tr}}$, $\|f(Q^\flat)\|_{\mathrm{tr}}$ are $\mathcal{O}(h^{-n})$, so we conclude that

$$\|f(P^\sharp) - f(P^\flat)\|_{\mathrm{tr}} = \mathcal{O}(h^{-n}). \quad (7.22)$$

Let $I_1 \subset\subset I_2 \subset\subset I_3 \subset\subset \mathbf{R}$ be fixed open intervals. Choosing $f \in C_0^\infty(\mathbf{R})$ first with $1_{I_1} \prec f \prec 1_{I_2}$ and then with $1_{I_2} \prec f \prec 1_{I_3}$, we get

$$N(I_1, P^\flat) - \mathcal{O}(h^{-n}) \leq N(I_2, P^\sharp) \leq N(I_3, P^\flat) + \mathcal{O}(h^{-n}). \quad (7.23)$$

It is now clear that if $P^\sharp$ satisfies (7.12) with $\lambda = 2$, then $P^\flat$ does so with $\lambda = 1$.

More generally, assume that $P^\sharp$ satisfies (7.12) for all $\lambda \geq 1$. Then we can view $\lambda^{-1}P^\sharp$ as a reference operator associated to $\lambda^{-1}P$, and all assumptions above, including (7.12) are satisfied uniformly with h replaced by $\widetilde{h} = h/\sqrt{\lambda}$. The previous argument gives (7.12) for $\lambda^{-1}P^\flat$ with h replaced by $\widetilde{h}$ and with λ replaced by 1/2. Making the inverse substitutions, we recover (7.12) for $P^\flat$. #

7.2 Analytic distorsions.

This classical approach to resonances was initiated by Aguilar-Combes [1], Balslev-Combes [5], and then followed by many others, [70, 43]. Here we recall the approach of [80] with some minor modifications of [76], and we refer to these papers for more details and references.

A smooth (C^∞) manifold $\Gamma \subset \mathbf{C}^n$ is called totally real if

$$T_x\Gamma \cap i(T_x\Gamma) = 0,\ \forall x \in \Gamma. \tag{7.24}$$

If Γ is totally real, then $\dim \Gamma \leq n$, and a natural example of a maximally totally real (m.t.r.) manifold (i.e. of maximal dimension n) is $\Gamma = \mathbf{R}^n$. (The opposite extreme of totally real manifolds is given by complex manifolds, for which $T_x\Gamma = iT_x\Gamma$.) Totally real manifolds are mapped to totally real manifolds under holomorphic diffeomorphisms, so the notion extends to submanifolds of complex manifolds.

Let $\Gamma \subset \mathbf{C}^n$ be smooth of real dimension n. Locally we have $\Gamma = f(\mathbf{R}^n)$, where $f : \mathbf{R}^n \to \mathbf{C}^n$ is smooth with injective differential. Let $\widetilde{f} : \mathbf{C}^n \to \mathbf{C}^n$ be an almost analytic extension of f so that $\overline{\partial}\widetilde{f}$ vanishes to infinite order on $\mathbf{R}^n$. Let $y \in \mathbf{R}^n$. Then since $d\widetilde{f}$ is complex linear, $iT_{f(y)}\Gamma = (d\widetilde{f}(y))(iT_y\mathbf{R}^n)$. Hence Γ is totally real near $f(y)$ iff

$$d\widetilde{f}(y)(T_y\mathbf{R}^n) \cap d\widetilde{f}(y)(iT_y\mathbf{R}^n) = 0,$$

i.e. iff $d\widetilde{f}$ is injective, or more explicitly iff

$$\det\left(\frac{\partial f_j(y)}{\partial y_k}\right) \neq 0 \tag{7.25}$$

If Γ is m.t.r. and $u \in C^\infty(\Gamma)$, then locally we can find an almost analytic extension $\widetilde{u} \in C^\infty(\mathbf{C}^n)$ with $\widetilde{u}_{|\Gamma} = u$, $\overline{\partial}\widetilde{u} = 0$ to infinite order on Γ. If $\widehat{u}$ is a second a.a. extension of the same function u, then $\widetilde{u}-\widehat{u}$ vanishes to ∞ order on Γ. In fact, let f be as above and let $\widetilde{f}$ be an a.a. extension, so that $\widetilde{f}$ is a local diffeomorphism according to (7.25). Then the inverse of f is a.a. on Γ and we can relate a.a. extensions from $\mathbf{R}^n$ and from Γ (locally) by means of $\widetilde{f}$ and its inverse.

Let $\Omega \subset \mathbf{C}^n$ be open, $\Gamma \subset \Omega$ a m.t.r. manifold. Let

$$P(x, D_x) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha \tag{7.26}$$

be an differential operator with holomorphic coefficient on Ω. We can define $P_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ by

$$P_\Gamma u = (P(x, D_x)\widetilde{u})_{|\Gamma}, \tag{7.27}$$

where $\widetilde{u}$ is a local a.a. extension of u. If f is as above then

$$(P_\Gamma u) \circ f = Q(y, D_y)(u \circ f), \tag{7.28}$$

$$Q(y, D_y) = \sum_{|\alpha|\le m} (a_\alpha \circ f)(y)(({}^t\partial_y \widetilde{f})^{-1} D_y)^\alpha. \tag{7.29}$$

In particular P_Γ is a differential operator of order m with smooth coefficients. The principal symbol q of Q is related to the principal symbol p of P by the usual relation

$$q(y, \eta) = p(f(y), ({}^t\partial \widetilde{f}(y))^{-1}\eta).$$

More invariantly, if we identify $T^*\Gamma$ with a submanifold of $\mathbf{C}^n \times \mathbf{C}^n$ via the map

$$T^*\Gamma \ni (x, d\phi(x)) \mapsto (x, \partial_x \widetilde{\phi}(x)) \in \Gamma^n \times \mathbf{C}^n,$$

for $\phi \in C^\infty(\mathbf{C}; \mathbf{R})$, then the principal symbol p_Γ of P_Γ is given by

$$p_\Gamma = p_{|T^*\Gamma}. \tag{7.30}$$

The quickest way to see this is to consider

$$p_\Gamma(x, d\phi(x)) = \lim_{h\to 0} e^{-i\phi(x)/h} P_\Gamma(e^{i\phi(x)/h}).$$

This remark also carries over to principal symbols in the sense of h-differential operators: Let

$$P(x, hD_x) = \sum_{|\alpha| \le m} a_\alpha(x; h)(hD_x)^\alpha$$

with a_α holomorphic in Ω, $a_\alpha = a_\alpha^0(x) + \mathcal{O}(h)$, and define the corresponding principal symbol by

$$p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha^0(x)\xi^\alpha.$$

Correspondingly we can consider h-differential operators on manifolds and define their principal symbols in the sense of h-differential operators. (7.30) still holds.

We need a deformation result.

Lemma 7.2 *Let $\omega \subset \mathbf{R}^n$ be open and let $f : [0, 1] \times \omega \ni (t, y) \mapsto f(t, y) \in \mathbf{C}^n$ be a smooth proper map such that $\det(\frac{\partial f}{\partial y}) \ne 0$, $\forall (t, y)$ and such that $f(t, \cdot)$ is injective. Assume that $f(t, y) = f(0, y)$ for all $y \in \omega \setminus K$, where K is some compact subset of ω. Let $P(x, D_x)$ be a differential operator with holomorphic coefficients defined in a neighborhood of $f([0, 1] \times \omega)$ such that P_{Γ_t} is elliptic for $0 \le t \le 1$, where $\Gamma_t = f(\{t\} \times \omega)$. If $u_0 \in \mathcal{D}'(\Gamma_0)$ and $P_{\Gamma_0} u_0$ extends to a holomorphic function in a neighborhood of $f([0, 1] \times \omega)$, then u_0 extends to a possibly multivalued holomorphic function in a neighborhood of the same set. More precisely, for every $t \in [0, 1]$, there is a holomorphic function u_t defined in a neighborhood of Γ_t such that $u_t = u_s$ near Γ_s when $|t-s|$ is small enough, and u_0 is an extension of u.*

This result is standard and was proved in [80]. We follow that work:

Proof. We shall use an FBI type argument. Let Γ be a m.t.r. manifold. We use Lebeau's resolution of the identity [51]: When $\Gamma = \mathbf{R}^n$, we can represent the Dirac measure δ as

$$\delta(x) = \int e^{i(x\cdot\xi + i\lambda|\xi|x^2/2)} b_\lambda(x, \xi) d\xi, \tag{7.31}$$

where $\lambda > 0$ and $b_\lambda(x, \xi) = \frac{1}{(2\pi)^n}(1 + \frac{i\lambda x\cdot\xi}{2|\xi|})$. This integral as well as the similar ones below should be interpreted as oscillatory integrals, i.e. as the limit in $\mathcal{D}'$, when $\epsilon \to 0$ of

$$\delta_\epsilon(x) = \int e^{i(x\cdot\xi + i\lambda|\xi|x^2/2)} b_\lambda(x, \xi) e^{-\epsilon\xi^2/2} d\xi,$$

and where the precise expression of the convergence factor $\exp(-\epsilon\xi^2/2)$ is not of great importance, since it suffices to have a family $\chi_\epsilon(\xi)$ of functions in $\mathcal{S}$ which is bounded in C_b^∞ and converges to 1 uniformly on every compact set when $\epsilon \to 0$. This means that if $u \in C_0^\infty(\mathbf{R}^n)$:

$$u(x) = \iint e^{i((x-y)\cdot\xi + i\lambda|\xi|(x-y)^2/2)} b_\lambda(x-y,\xi)u(y)dyd\xi. \tag{7.32}$$

Put $\phi_\alpha(x,y,\alpha) = (x-y)\cdot\alpha_\xi + i\lambda|\alpha_\xi|((x-\alpha_x)^2 + (y-\alpha_x)^2)$, $a_\lambda(x,y,\alpha) = c(\lambda)|\alpha_\xi|^{\frac{n}{2}} b(x-y,\alpha_\xi)$ for a suitable $c(\lambda) > 0$, $\alpha = (\alpha_x,\alpha_\xi)$. Then by evaluating a Gaussian integral in α_x, we get from (7.32) :

$$u(x) = \iint_{\substack{y\in\mathbf{R}^n \\ \alpha\in\mathbf{R}^{2n}}} e^{i\phi_\lambda(x,y,\alpha)} a_\lambda(x,y,\alpha)u(y)dyd\alpha. \tag{7.33}$$

We now pass to the case of general Γ, replacing $|\alpha_\xi|$ by its holomorphic extension $\sqrt{\alpha_\xi^2}$, and work near some fixed point $x_0 \in \Gamma$. We may assume $x_0 = 0$, $T_{x_0}^*\Gamma = \mathbf{R}^n$. With $\lambda > 0$ sufficiently large, $\phi = \phi_\lambda$, $a = a_\lambda$, we put

$$Au(x) = \iint_{\Gamma\times T^*\Gamma} e^{i\phi(x,y,\alpha)} a(x,y,\alpha)\chi_1(\alpha_x)\chi(y)u(y)dyd\alpha,\ x\in\Gamma, \tag{7.34}$$

where $\chi \in C_0^\infty(\Gamma)$ is equal to 1 near 0 and $\chi \prec \chi_1 \in C_0^\infty(\Gamma)$. Notice that

$$\operatorname{Im}\phi \sim |\alpha_\xi|(|x-\alpha_x|^2 + |y-\alpha_x|^2).$$

By Stokes' formula, we can change the integration contour to $y \in \Gamma$, $\alpha_x \in \Gamma$, $\alpha_\xi \in T_y^*(\Gamma)$. (This is a change in α_ξ only, and we can use the intermediate contours given by $y, \alpha_x \in \Gamma$, $\alpha_\xi \in T^*_{[ty+(1-t)\alpha_x]}\Gamma$, where $[z]$ denotes the point in Γ which is closest to z, for z in a small complex neighborhood of x_0.) Then we can integrate out α_x and get

$$Au(x) \equiv \int_{y\in\Gamma}\int_{\xi\in T_y^*\Gamma} e^{i((x-y)\cdot\xi + i\lambda|\xi|(x-y)^2/2)} b_\lambda(x-y,\xi)\chi(y)u(y)dyd\xi =: \widetilde{A}u(x), \tag{7.35}$$

modulo a term $Ku(x)$, which extends holomorphically to a u-independent neighborhood of x_0. By analytic continuation from (7.32) we see that the distribution kernel of $\widetilde{A}$ vanishes outside the diagonal and since $\widetilde{A}$ is a pseudodifferential operator of order 0, we have

$$\widetilde{A}u = \widetilde{a}(x)u(x). \tag{7.36}$$

Testing $\widetilde{A}$ on oscillatory functions and using Stokes' formula, we see that $\widetilde{a} = \chi(x)$, so we end up with

$$\chi u(x) \equiv Au(x). \tag{7.37}$$

Using the ellipticity of P, we can construct an analytic symbol $c(x, y, \alpha)$, with α_ξ as the large parameter (see for instance [73]) of order $\frac{n}{2} - m$, defined in a conic neighborhood of $\{x_0, x_0\} \times T^*_{x_0}\Gamma$, holomorphic in x, y, α_x and with $\overline{\partial}_{\alpha_\xi} c$ exponentially small ($\mathcal{O}(e^{-|\alpha_\xi|/C})$ for some $C > 0$), such that

$${}^tP(y, D_y)(e^{i\phi(x,y,\alpha)}c(x, y, \alpha)) = e^{i\phi(x,y,\alpha)}(a(x, y, \alpha) - r(x, y, \alpha)), \tag{7.38}$$

where r is holomorphic in x, y and of exponential decrease:

$$|r(x, y, \alpha)| \leq Ce^{-|\alpha_\xi|/C}. \tag{7.39}$$

Using this, we get for x in a neighborhood of x_0 and assuming $\operatorname{supp}\chi$ small:

$$\begin{aligned} Au(x) = &\iint e^{i\phi(x,y,\alpha)}r(x, y, \alpha)\chi_1(\alpha_x)(\chi u)(y)dyd\alpha \\ &+ \iint e^{i\phi(x,y,\alpha)}c(x, y, \alpha)\chi(y)\chi_1(\alpha_x)Pu(y)dyd\alpha \\ &+ \iint e^{i\phi(x,y,\alpha)}c(x, y, \alpha)\chi_1(\alpha_x)[P, \chi]u(y)dyd\alpha. \end{aligned} \tag{7.40}$$

Notice that $({}^tP)_\Gamma = {}^t(P_\Gamma)$.

The first and the last integrals extend holomorphically to a u-independent neighborhood of x_0. For the middle integral, we get the same conclusion, if we assume that Pu extends to some fixed neighborhood Ω of x_0. To see this we notice that the exponential factor $e^{i\phi(x,y,\alpha)}$ become exponentially small, if we make a small α-dependent deformation of the integration in y near $y = x_0$ and restrict x to sa small complex neighborhood of that point.

Summing up, we have proved: Let P be holomorphic near Γ with P_Γ elliptic. If Γ is m.t.r., $x_0 \in \Gamma$, $u \in \mathcal{D}'(\Gamma)$ and Pu extends to some neighborhood Ω of x_0, then u extends to another neighborhood $\widetilde{\Omega}$ of x_0, which does not depend on u. Also $\widetilde{\Omega}$ will not collapse, if we vary Γ reasonably. From this the lemma follows #

We now return to the operators in section 7.1 and add an assumption:

$$\begin{aligned}&\text{There exist } \theta_0 \in]0,\pi[, \epsilon > 0, R_1 \geq R_0, \text{ such that the}\\&\text{coefficients } a_\alpha(x;h) \text{ of } Q \text{ extend holomorphically in } x \text{ to}\\&\{r\omega;\, \omega\in \mathbf{C}^n,\, \mathrm{dist}\,(\omega, S^{n-1})<\epsilon,\, r\in e^{i[0,\theta_0]}]R_1,+\infty[\},\\&\text{and the relevant parts of (7.6), (7.8) remain valid}\end{aligned} \tag{7.41}$$

in the sense that a_α and all its x-derivatives are bounded for x in this larger set, uniformly in h, and the convergence in (7.8) remains valid when x tends to infinity in the larger set above, again uniformly in h.

For given $\epsilon_0 > 0$, $R_1 > R_0$, we can construct a smooth function $[0,\pi]\times[0,\infty[\ni(\theta,t)\mapsto f_\theta(t)\in\mathbf{C}$, injective for every θ, with the properties

(i) $f_\theta(t) = t$ for $0\leq t\leq R_1$,

(ii) $0\leq \arg f_\theta(t)\leq\theta$, $\partial_t f_\theta\neq 0$,

(iii) $\arg f_\theta(t)\leq \arg\partial_t f_\theta(t)\leq \arg f_\theta(t)+\epsilon_0$,

(iv) $f_\theta(t) = e^{i\theta}t$, for $t\geq T_0$, where T_0 only depends on ϵ_0 and R_1.

Consider the map

$$\kappa_\theta : \mathbf{R}^n\ni x = t\omega\mapsto f_\theta(t)\omega\in\mathbf{C}^n,\ \ t = |x|.$$

The image Γ_θ is a m.t.r. manifold which coincides with $\mathbf{R}^n$ along $B(0,R_1)$. We choose R_1 at least as large as R_1 in the assumption (7.41).

Let $\mathcal{H}_\theta = \mathcal{H}_{R_0}\oplus L^2(\Gamma_\theta\setminus B(0,R_0))$. By means of κ_θ we can identify $\mathcal{H}_\theta$ with $\mathcal{H}$ and similarly we can define $\mathcal{D}_\theta = \mathcal{D}$. Let $\chi\in C_0^\infty(B(0,R_1))$ be equal to 1 near $\overline{B(0,R_0)}$ and define the unbounded operator $P_\theta : \mathcal{H}\to\mathcal{H}$ with domain $\mathcal{D}$ by

$$P_\theta u = P(\chi u) + Q_{\Gamma_\theta}(1-\chi)u.$$

Here we assume that $0\leq\theta\leq\theta_0$.

Parametrizing Γ_θ by means of κ_θ we get:

$$-\Delta_{\Gamma_\theta} = (\frac{1}{f'(t)}D_t)^2 - \frac{n-1}{f(t)f'(t)}iD_t + (f(t))^{-2}D_\omega^2,\ f = f_\theta,$$

where $-D_\omega^2$ denotes the Laplacian on S^{n-1}. If ω^{*2} denotes the principal symbol of D_ω^2 and τ is the dual variable of t, then the principal symbol of $-\Delta_{\Gamma_\theta}$ is

$$p_{0,\theta} = \frac{\tau^2}{f'(t)^2} + \frac{\omega^{*2}}{f(t)^2}, \tag{7.42}$$

so pointwise $-\Delta_{\Gamma_\theta}$ is elliptic (if $\epsilon_0 > 0$ is small enough) and the principal symbol $p_{0,\theta}$ takes its values in a sector of angle $\leq 2\epsilon_0$ and globally it takes its values in the sector

$$-2(\theta + \epsilon_0) \leq \arg p_{0,\theta} \leq 0. \tag{7.43}$$

Since $\theta \leq \theta_0$, this sector is of angle $< 2\pi$ (when ϵ_0 is small enough).

Choosing R_1 large enough, we get

In $\mathbf{R}^n \setminus B(0, R_0)$, $h^{-2}P_\theta$ is an elliptic differential operator whose principal symbol in the classical sense over each fixed point in Γ_θ takes its values in a sector of angle $\leq 3\epsilon_0$, and globally in a sector $-2\theta - 3\epsilon_0 \leq \arg z \leq \epsilon_0$ (7.44)

The coefficients of $P_\theta - e^{-2i\theta}(-h^2\Delta_{\mathbf{R}^n})$ (as a semiclassical differential operator) tend to 0 uniformly w.r.t. h, when $\Gamma_\theta \ni x \to \infty$, and we identify Γ_θ and $\mathbf{R}^n$ by means of κ_θ. (7.45)

Lemma 7.3 *If $z \in \mathbf{C}\setminus\{0\}$, $\arg z \neq -2\theta$, then $P_\theta - z : \mathcal{D} \to \mathcal{H}$ is a Fredholm operator of index* 0.

The proof [75] is a minor modification of the corresponding one in [80]:

Proof. This has nothing to do with the smallness of h, so we assume $h = 1$ for simplicity. We first show that $P_\theta - z$ is a Fredholm operator, and for that it suffices to invert $P_\theta - z$ modulo compact operators. On $\Gamma_\theta \simeq \mathbf{R}^n$, we introduce a smooth partition of unity :

$$1 = \theta_1 + \theta_2 + \theta_3, \tag{7.46}$$

where $1_{B(0,R_0)} \prec \theta_1 \in C_0^\infty(B(0, R_1))$, $\theta_2 \in C_0^\infty$, $\operatorname{supp}\theta_3 \subset \{x \in \mathbf{R}^n; |x| \geq R\}$, $R \gg 1$. Further, $0 \leq \theta_j \leq 1$. Let χ_1, χ_2, χ_3 have the same properties as θ_j with the exception of (7.46) and with $\theta_j \prec \chi_j$. Let $z_0 \in \mathbf{C} \setminus \mathbf{R}$, and put

$$E(z) = \chi_1(P - z_0)^{-1}\theta_1 + Q\widetilde{Q}\theta_2 + \chi_3(-e^{-2i\theta}\Delta - z)^{-1}\theta_3,$$

where $\widetilde{Q}$ is a properly supported parametrix of $P_\theta - z$ (i.e. a pseudodifferential operator which inverts the latter operator modulo smoothing operators outside a small neighborhood of the closure of $B(0, R_0)$ with the property that the distribution kernel is supported in a thin neighborhood of the diagonal). Here θ_3, χ_3 have their support in the region where $\Gamma_\theta = e^{i\theta}\mathbf{R}^n$. Then we get

$$(P_\theta - z)E(z) = 1 + K(z),$$

with

$$\begin{aligned} K(z) = & \ (z_0 - z)\chi_j(P - z_0)^{-1}\theta_1 + [P_\theta, \chi_1](P - z_0)^{-1}\theta_1 + ((P_\theta - z)\widetilde{Q} - 1)\theta_2 \\ & + [P_\theta, \chi_3](-e^{-2i\theta}\Delta - z)^{-1}\theta_3 + \chi_3(P_\theta - (-e^{-2i\theta}\Delta))(-e^{-2i\theta}\Delta - z)^{-1}\theta_3 \\ = & \ K'(z) + K''(z), \end{aligned}$$

where $K''(z)$ is the last term in the middle member. With R large enough, we can arrange so that $\|K''(z)\|_{\mathcal{L}(L^2,L^2)} < 1/2$. On the other hand, $K'(z)$ is compact. Now write

$$(P_\theta - z)E(z)(1 + K''(z))^{-1} = 1 + K'(1 + K'')^{-1},$$

to get a right inverse modulo a compact operator.

As an approximate left inverse, we put

$$F(z) = \theta_1(P - z_0)^{-1}\chi_1 + \theta_2\widetilde{Q} + \theta_3(-e^{-2i\theta}\Delta - z)^{-1}\chi_3.$$

Then

$$F(P_\theta - z) = 1 + L(z), \ L(z) = L'(z) + L''(z), \tag{7.47}$$

where

$$\begin{aligned} L'(z) &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \\ L'' &= \theta_3(-e^{-2i\theta}\Delta - z)^{-1}(P_\theta - (-e^{-2i\theta}\Delta))\chi_3, \\ \mathrm{I} &= (z_0 - z)\theta_1(P - z_0)^{-1}\chi_1, \\ \mathrm{II} &= -\theta_1(P - z_0)^{-1}[P, \chi_1], \\ \mathrm{III} &= \theta_2(\widetilde{Q}(P_\theta - z) - 1), \\ \mathrm{IV} &= -\theta_3(-e^{-2i\theta}\Delta - z)^{-1}[P_\theta, \chi_3]. \end{aligned}$$

We can arrange so that $\|L''\|_{\mathcal{L}(\mathcal{D}_\theta,\mathcal{D}_\theta)}$ is a small as we like, and we turn to the proof that L' is compact: $\mathcal{D}_\theta \to \mathcal{D}_\theta$.

Clearly, III, IV have this compactness property. For I and II, we first see that they are bounded: $\mathcal{D}_\theta \to \mathcal{D}_\theta$, so it is enough to show that $(P-z_0)\mathrm{I}$, $(P-z_0)\mathrm{II}$ are compact: $\mathcal{D}_\theta \to \mathcal{H}_\theta$. We have

$$(P-z_0)\mathrm{I} = (z-z_0)\theta_1 + (z_0-z)[P,\theta_1](P-z_0)^{-1}\chi_1,$$

where θ_1 is compact: $\mathcal{D} \to \mathcal{H}$, and $[P,\theta_1](P-z_0)^{-1}\chi_1$ is even compact as an operator : $\mathcal{H} \to \mathcal{H}$. Similarly,

$$(P-z_0)\mathrm{II} = \underbrace{\theta_1[P,\chi_1]}_{=0} - \underbrace{[P,\theta_1](P-z_0)^{-1}[P,\chi_1]}_{\text{compact:}\mathcal{D}\to\mathcal{H}}$$

is compact. Hence L' is compact $\mathcal{D}_\theta \to \mathcal{D}_\theta$ and we invert $(P_\theta - z)$ to the left modulo a compact operator $\mathcal{D}_\theta \to \mathcal{D}_\theta$, by means of $(1+L'')^{-1}F$:

$$(1+L'')^{-1}F(P_\theta - z) = 1 + (1+L'')^{-1}L'.$$

We have then showed that $P_\theta - z : \mathcal{D}_\theta \to \mathcal{H}_\theta$ is Fredholm. This operator depends contiuously on (θ, z), so the index is constant under deformation in (θ, z) with $z \notin e^{-2i\theta}[0,\infty[$. Deforming to $\theta = 0$, $z = i$, we see that the index is 0. #

Corollary 7.4 *A point z in $\mathbf{C}\backslash e^{-2i\theta}[0,\infty[$ belongs to $\sigma(P_\theta)$ iff $\mathcal{N}(P_\theta - z) \neq 0$.*

Proposition 7.5 *Assume that $0 \le \theta_1 < \theta_2 \le \theta_0$, and let $z_0 \in \mathbf{C}\backslash e^{-2i[\theta_1,\theta_2]}[0,\infty[$. Then*

$$\dim \mathcal{N}(P_{\theta_2} - z_0) = \dim \mathcal{N}(P_{\theta_1} - z_0).$$

Proof. We follow [80] ([75]). Again the statement does not depend on the smallness of h, so we take $h = 1$ for simplicity. We shall prove that the two kernels are "equal" via holomorphic extension. We shall apply Lemma 7.2 to suitable deformations of the contours. Let $\chi \in C_0^\infty(]\frac{1}{2}, \frac{3}{2}[; [0,1])$ be equal to 1 near 1 and put for $T \ge 1$:

$$F_{\theta_1,\theta_1,T}(t) = f_{\theta_1}(t) + \chi(\frac{t}{T})(f_{\theta_2}(t) - f_{\theta_1}(t)).$$

Let $\Gamma_{\theta_1,\theta_2,T}$ be the corresponding submanifold of $\mathbf{C}^n$. If $\theta_2 - \theta_1$ is small enough, Lemma 7.2 is applicable to the family of m.t.r. manifolds,

$$[0,1] \ni t \mapsto \Gamma_{\theta_1,\theta_1+t(\theta_2-\theta_1),T}.$$

We conclude that if $(P_{\theta_1} - z_0)u_{\theta_1} = 0$, $u \in \mathcal{D}_{\theta_1}$, then u_{θ_1} extends to a holomorphic function in a neighborhood of $\cup_{\theta_1 \le \theta \le \theta_2}(\Gamma_\theta \setminus B(0, R_1))$, and by restriction, we get an element $u_\theta \in \mathcal{D}_\theta^{\mathrm{loc}}$ with $(P_\theta - z_0)u_\theta = 0$.

We also need to control u_θ near infinity, and for that we shall use the assumption on z_0 which assures ellipticity near ∞.

Let $0 < a_1 < a_2 < b_2 < b_1$, $\Omega_j = \{z \in \mathbf{R}^n;\ a_j < |z| < b_j\}$, $j = 1, 2$. The change of variables $y = Tz$ transforms $-\Delta_y$ into $-\frac{1}{T^2}\Delta_z$. We take $T \ge 1$ and view the latter operator as semi-classical one with $h = 1/T$, and use the standard a priori estimate

$$\sum_{|\alpha| \le 2} \|(T^{-1}D_z)^\alpha u\|_{L^2(\Omega_2)} \le C(\|(-T^{-2}e^{-2i\theta}\Delta_z - z_0)u\|_{L^2(\omega_1)} + \|u\|_{L^2(\Omega_1 \setminus \Omega_2)}),$$

with C independent of T. In the y-variables this becomes

$$\sum_{|\alpha| \le 2} \|D_y^\alpha u\|_{L^2(T\Omega_2)} \le C(\|(-e^{-2i\theta}\Delta_y - z_0)u\|_{L^2(T\Omega_1)} + \|u\|_{L^2(T(\Omega_1 \setminus \Omega_2))}). \quad (7.48)$$

For $\theta_2 - \theta_1$ small enough and T large enough, and for t/T near $\operatorname{supp}\chi$, we may view $P_{\Gamma_{\theta_1,\theta_2,T}}$ as a small perturbation of $-e^{-2i\theta_1}\Delta_y$. We conclude that

$$\|u_{\theta_2}\|_{H^2(T\Omega)} \le C\|u_{\theta_1}\|_{L^2(TV)}.$$

Now cover a neighborhood of ∞ in Γ_{θ_2} by a sequence $T_k\Omega$, where $T_k = 2^k T_0$. If $u_{\theta_1} \in \mathcal{D}_{\theta_1}$, it follows that $u_{\theta_2} \in \mathcal{D}_{\theta_2}$.

To sum up, if $\theta_2 - \theta_1$ is small enough, we have constructed an injective linear map (by holomorphic extension and restriction): $\mathcal{N}(P_{\theta_1} - z_0) \to \mathcal{N}(P_{\theta_2} - z_0)$ The inverse map is constructed the same way, so $\dim \mathcal{N}(P_{\theta_1} - z_0) = \dim \mathcal{N}(P_{\theta_2} - z_0)$.

When $\theta_2 - \theta_1$ is larger, we introduce intermediate θ-values and apply the above. #

Two extensions of the lemma are possible:

1) We may consider inhomogeneous equations for $(P_\theta - z)$ assuming suitable holomorphic extension properties of the right hand side.

2) We may consider $\mathcal{N}((P_\theta - z)^k)$ for $k \in \mathbf{N}$.

The preceding result shows that for $0 \le \theta_1 \le \theta \le \theta_2 \le \theta_0$, the spectrum of P_θ in $\mathbf{C} \setminus e^{-2i[\theta_1,\theta_2]}[0, +\infty[$ is independent of θ. In particular P_θ will have no spectrum in a small sector $e^{i]0,\epsilon[}]0, +\infty[$. Analytic Fredholm theory shows that the spectrum of P_θ in $\mathbf{C} \setminus e^{-2i\theta}[0, +\infty[$ is discrete. In particular for

$\theta = 0$, we see that the spectrum in $]-\infty, 0[$ is discrete. If $0 < \theta < \frac{\pi}{2}$, the spectrum of P_θ outside $e^{-2i\theta}[0, +\infty[$ consists of the negative eigenvalues of P_0 and the eigenvalues in $e^{-i[0,2\theta[}]0, +\infty[$. When $\theta \geq \frac{\pi}{2}$, we only get the latter eigenvalues. By definition, the resonances of P are the eigenvalues of P_θ in the sector $e^{-2i[0,\theta[}]0, +\infty[$, and as we have seen, they are independent of θ in the sense that replacing θ by a larger value will not change the set of resonances in the sector $e^{-2i[0,\theta[}]0, +\infty[$.

By analytic Fredholm theory, if $z_0 \in e^{-i[0,2\theta[}]0, +\infty[$, $0 \leq \theta \leq \theta_0$ is a resonance, then the spectral projection

$$\pi_{\theta,z_0} = \frac{1}{2\pi i} \int_{\gamma(z_0)} (z - P_\theta)^{-1} dz \tag{7.49}$$

is of finite rank, where $\gamma(z_0)$ denotes the positively oriented boundary of a small disc centered at z_0, which contains no other resonances than z_0. This rank is independent of the choice of θ for which z_0 belongs to the sector where the resonances are defined, and by definition it is the multiplicity of the resonance z_0. The image F_{θ,z_0} is contained in the domain of any power of P and is invariant under P_θ. $(P_\theta - z_0)_{|F_{\theta,z_0}}$ is nilpotent and $F_{\theta,z_0} = \mathcal{N}((P_\theta - z_0)^{k_0})$ for some $k_0 \in \mathbf{N}$. The proof of Proposition 2.4 gives a bijection $F_{\theta_1,z_0} \to F_{\theta_2,z_0}$ under the assumptions there.

To end this chapter we mention that resonances defined by means of complex distorsions, coincide with other definitions. (See [37] for general results in this direction.) In particular, one can show that the resolvent $(z - P)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$, defined first for $\operatorname{Im} z > 0$, can be extended meromorphically across $]0, +\infty[$ to the sector $e^{-i[0,2\theta_0[}]0, +\infty[$ and the poles are just the resonances. [[[Add the proof of this statement]]]. The multiplicity of a resonance is the rank of the formal spectral projection, obtained by replacing the resolvent of P_θ in the integral (7.49) by the meromorphic extension of the resolvent of P. See [80].

The set of resonances will be denoted by $\operatorname{Res} P$ and the elements of this set will be counted (repeated) according to their multiplicity.

8 The local trace formula and local upper bounds

8.1 Some trace estimates

Let $P_\cdot : \mathcal{H} \to \mathcal{H}$, $\cdot = 0, 1$, be two selfadjoint operators satisfying the assumptions in the preceding section. Here

$$\mathcal{H}_\cdot = \mathcal{H}_{\cdot,R_0} \oplus L^2(\mathbf{R}^n \setminus B(0, R_0)) \tag{8.1}$$

is an orthogonal sum. In particular for the reference operators $P_\cdot^\sharp$, we have corresponding estimates $\#(\sigma(P_\cdot) \cap [-\lambda, \lambda]) \leq \mathcal{O}(1)\Phi_\cdot(\frac{\lambda}{h^2})$. In view of (7.13), we notice that $\Phi_\cdot$ are of at most polynomial growth:

$$\Phi_\cdot(t) = \mathcal{O}(t^{n_\cdot/2}) \tag{8.2}$$

for some exponents $n_\cdot \in [n, +\infty[$.

To the earlier assumptions, we add:

$$|[a_{\cdot,\alpha}(x; h)]_0^1| \leq \mathcal{O}(1)\langle x\rangle^{-\widetilde{n}}, \text{ for some } \widetilde{n} > n \tag{8.3}$$

not only in the real domain but also in the complex domain appearing in (7.41). Here $\theta_0 > 0$ is assumed to be the same for both operators P_1, P_0, and we write $[a_\cdot]_0^1 = a_1 - a_0$.

Proposition 8.1 *Let $f \in C_0^\infty(\mathbf{R})$ be independent of h or vary in a bounded subset of $C_0^\infty(\mathbf{R})$. Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be equal to 1 near $\overline{B(0, R_0)}$. Then $\chi f(P_\cdot)$, $f(P_\cdot)\chi$, $[(1-\chi)f(P_\cdot)(1-\chi)]_0^1$ are of trace class and*

$$\begin{aligned} &''\mathrm{tr}\,[f(P_\cdot)]_0^1{}'' := &(8.4)\\ &[\mathrm{tr}\,(\chi f(P_\cdot)\chi + \chi f(P_\cdot)(1-\chi) + (1-\chi)f(P_\cdot)\chi)]_0^1 + \mathrm{tr}\,[(1-\chi)f(P_\cdot)(1-\chi)]_0^1 \end{aligned}$$

is independent of the choice of χ and is $\mathcal{O}(1)\Phi_{\max}(h^{-2})$. Here $\Phi_{\max} := \max(\Phi_0, \Phi_1)$.

Let $\epsilon_0 > 0$ be fixed and small enough, so that $2\theta_0 + \epsilon_0 < 2\pi$. Let $W \subset\subset \Omega$ be open relatively compact subsets of $e^{i]-2\theta_0,\epsilon_0]}]0, +\infty[$. We assume that these sets are independent of h and that Ω is simply connected. Also assume that the intersections J_+, I_+ between Ω, W and $]0, +\infty[$ are intervals. Let Ω_-, W_- denote the intersections of $e^{i]-2\theta_0,0]}]0, +\infty[$ with Ω and W respectively. The local trace formula ([75, 76]) is given in the following result:

Theorem 8.2 *Fix Ω, W as above. Then for $h > 0$ sufficiently small the following holds: Let $f = f(z;h)$ be holomorphic for $z \in \Omega$ and satisfy $|f(z;h)| \leq 1$, for $z \in \Omega \setminus W$. Let $\chi \in C_0^\infty(J_+)$ be independent of h, with $\chi = 1$ near $\overline{I}_+$. Then uniformly with respect to f:*

$$
\begin{aligned}
&{}''\mathrm{tr}\,[(\chi f)(P_\cdot)]_0^{1}{}'' = \\
&[\sum_{\lambda \in \mathrm{Res}\, P_\cdot \cap W_-} f(\lambda)]_0^1 - [\sum_{\mu \in \sigma(P_\cdot) \cap]-\infty,0[\cap W_-} f(\mu)]_0^1 + \mathcal{O}(1)\Phi_{\max}(\tfrac{1}{h^2}).
\end{aligned}
\tag{8.5}
$$

The proof also shows that the number of resonances of $P_\cdot$ in Ω_- is $\mathcal{O}(1)\Phi_\cdot(h^{-2})$.

Later in these notes we will also discuss the more classical trace formula of Bardos-Guillot-Ralston [6] with an important improvement of R. Melrose [55] and further generalized in [85]. That formula relates all the resoances to the trace of the wave-group for operators as in Chapter 2, when the dimension is odd. In the cited works, it is proved by using the Lax–Phillips theory and in particular translation represenations, but following the work [35], Zworski also gave a proof in [102] which uses more general scattering theory and which is closer to the proof that we shall give for the local trace formula (following [76]).

It should also be pointed out that the left hand side in the local trace formula is the same as in the Birman–Krein formula for the scattering phase. See [...].

In this section we shall prove Proposition 8.1, by adapting arguments from [80] and the local trace formula will be proved in the next section. When working with the $P_\cdot$ separately, we generally drop the subscript "$\cdot$". If χ is as in the Proposition, choose a torus T for the definition of a reference operator $P^\sharp$ as in Chapter 7, and choose $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^n)$ with $\chi \prec \widetilde{\chi}$. We may arrange so that $\mathrm{supp}\,\widetilde{\chi}$ is naturally included in T and $P^\sharp = P$ near $\mathrm{supp}\widetilde{\chi}$. Recall that trace negligible operators were introduced in Chapter 7.

Lemma 8.3 *The following operators are trace negligible for z in any fixed bounded set in $\mathbf{C} \setminus \mathbf{R}$:*

$$
\begin{aligned}
&\chi(z-P)^{-1}\chi - \chi(z-P^\sharp)^{-1}\chi, \\
&(1-\chi)(z-P)^{-1}(1-\chi) - (1-\chi)(z-Q)^{-1}(1-\chi), \\
&\chi(z-P)^{-1}(1-\widetilde{\chi}), \\
&\chi(z-Q)^{-1}(1-\widetilde{\chi}).
\end{aligned}
\tag{8.6}
$$

Proof. We start with $\chi(z-P)^{-1}(1-\widetilde{\chi})$. Let $\chi \prec \theta_N \prec ... \prec \theta_1 \prec \widetilde{\chi}$, $\theta \in C_0^\infty$. Then we have the telescopic formula:

$$(z-P)^{-1}(1-\widetilde{\chi}) =$$

$$\sum_{j=1}^N (1-\theta_j)(z-Q)^{-1}[Q,1-\theta_{j-1}]...(z-Q)^{-1}[Q,1-\theta_1](z-Q)^{-1}(1-\widetilde{\chi})$$

$$+(z-P)^{-1}[Q,1-\theta_N](z-Q)^{-1}[Q,1-\theta_{N-1}]...[Q,1-\theta_1](z-Q)^{-1}(1-\widetilde{\chi}),$$

so

$$\chi(z-P)^{-1}(1-\widetilde{\chi}) =$$

$$\chi(z-P)^{-1}[Q,1-\theta_N](z-Q)^{-1}[Q,1-\theta_{N-1}]...[Q,1-\theta_1](z-Q)^{-1}(1-\widetilde{\chi}).$$

We conclude as in the proof of the invariance of (7.12), and see that $\chi(z-P)^{-1}(1-\widetilde{\chi})$ is trace negligible. The operator $\chi(z-Q)^{-1}(1-\widetilde{\chi})$ can be treated in the same way.

Let $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ be equal to one near $\overline{B(0,R_0)}$ and satisfy $\chi_0 \prec \chi$. Write

$$(z-P)^{-1}(1-\chi) = (1-\chi_0)(z-Q)^{-1}(1-\chi) - (z-P)^{-1}[Q,\chi_0](z-Q)^{-1}(1-\chi),$$

so that

$$(1-\chi)(z-P)^{-1}(1-\chi) - (1-\chi)(z-Q)^{-1}(1-\chi) =$$

$$-((1-\chi)(z-P)^{-1}\chi_1)([Q,\chi_0](z-Q)^{-1}(1-\chi)),$$

where we inserted a cutoff $\chi_1 \in C_0^\infty$ with $\chi_0 \prec \chi_1 \prec \chi$, and indicated a factorization of the last term into 2 factors. The first of the two factors is trace negligible, while the second one is bounded of norm $\mathcal{O}(|\mathrm{Im}\, z|^{-1})$, so the operator above is trace negligible.

The argument for $\chi(z-P)^{-1}\chi - \chi(z-P^\sharp)^{-1}\chi$ is the same except that we use $\widetilde{\chi}$ instead of $1-\chi_0$ and χ instead of $1-\chi$. #

Let $\chi_1 \in C_0^\infty(\mathbf{R}^n)$ with $\chi \prec \chi_1$, and choose the reference torus large enough, to include $\mathrm{supp}\, \chi_1$. Consider the short telescopic formula:

$$(z-P)^{-1}\chi = \chi_1(z-P^\sharp)^{-1}\chi + (z-P)^{-1}[P,\chi_1](z-P^\sharp)^{-1}\chi.$$

The proof of Lemma 8.3 shows that $[P,\chi_1](z-P^\sharp)^{-1}\chi$ is trace negligible, so the same holds for $(z-P)^{-1}[P,\chi_1](z-P^\sharp)^{-1}\chi$.

Let $f\in C_0^\infty(\mathbf{R})$. Then,

$$f(P)\chi=\chi_1 f(P^\sharp)\chi+\mathcal{O}(h^\infty)$$

in trace norm. The trace norm of $\chi_1 f(P^\sharp)\chi$ is $\mathcal{O}(1)\Phi(h^{-2})$, so the same holds for $f(P)\chi$ and by duality for $\chi f(P)$.

From the lemma it follows that the trace class norm of $(1-\chi)f(P)(1-\chi)-(1-\chi)f(Q)(1-\chi)$ is $\mathcal{O}(h^\infty)$, so in order to prove that the trace class norm of $[(1-\chi)f(P_\cdot)(1-\chi)]_0^1$ is $\mathcal{O}(h^{-n})$, as we shall, it suffices to prove the same fact about $[f(Q_\cdot)]_0^1$.

Let $1=\sum_{j\in\mathbf{Z}^n}\chi_j(x)$ be a partition of unity with $\chi_j(x)=\chi_0(x-j)\in C_0^\infty$. For $N\in\mathbf{N}$, consider

$$(i+Q_1)^{-N}(z-Q_1)^{-1}-(i+Q_0)^{-N}(z-Q_0)^{-1}, \tag{8.7}$$

which is a linear combination of terms

$$(i+Q_1)^{-k}(Q_1-Q_0)(i+Q_0)^{-(N+1-k)}(z-Q_0)^{-1} \tag{8.8}$$

for $1\le k\le N$, and of the term

$$(i+Q_1)^{-N}(z-Q_1)^{-1}(Q_1-Q_0)(z-Q_0)^{-1}. \tag{8.9}$$

For (8.8), we consider

$$\chi_\nu(i+Q_1)^{-k}(Q_1-Q_0)\chi_\rho(i+Q_0)^{-(N+1-k)}(z-Q_0)^{-1}\chi_\mu. \tag{8.10}$$

Using a Combes-Thomas argument, (see [66], [39], [75], section 7) [explain..] we see that this operator is bounded: $L^2\to H^{2(N+1)}$ of norm

$$\begin{aligned}&\mathcal{O}(1)\tfrac{1}{|\operatorname{Im}z|}\langle\rho\rangle^{-\widetilde{n}}e^{-\frac{1}{Ch}((|\nu-\rho|-C)_++|\operatorname{Im}z|(|\rho-\mu|-C)_+)},\\ &\text{with } t_+=\max(t,0),\ t\in\mathbf{R}.\end{aligned} \tag{8.11}$$

If $N\ge N(n)$ is large enough, this is also a bound for h^n times the trace norm for operators $L^2\to L^2$, and summing over ν, μ, we get

$$\begin{aligned}&\|(i+Q_1)^{-k}(Q_1-Q_0)\chi_\rho(i+Q_0)^{-(N+1-k)}(z-Q_0)^{-1}\|_{\mathrm{tr}}\\ &=\mathcal{O}(h^{-n})\langle\rho\rangle^{-\widetilde{n}}\tfrac{1}{|\operatorname{Im}z|}(1+\tfrac{h}{|\operatorname{Im}z|})^n.\end{aligned} \tag{8.12}$$

For (8.9) we get similarly

$$
\begin{aligned}
&\|(i+Q_1)^{-N}(z-Q_1)^{-1}(Q_1-Q_0)\chi_\rho(z-Q_0)^{-1}\|_{\mathrm{tr}} \\
&= \mathcal{O}(h^{-n})\langle\rho\rangle^{-\widetilde{n}}\tfrac{1}{|\mathrm{Im}\, z|^2}(1+\tfrac{h}{|\mathrm{Im}\, z|})^{2n}.
\end{aligned}
\tag{8.13}
$$

Summing over ρ, we get

$$
\begin{aligned}
&\|(i+Q_1)^{-N}(z-Q_1)^{-1}-(i+Q_0)^{-N}(z-Q_0)^{-1}\|_{\mathrm{tr}} \\
&= \mathcal{O}(h^{-n})\tfrac{1}{|\mathrm{Im}\, z|^2}(1+\tfrac{h}{|\mathrm{Im}\, z|})^{2n}.
\end{aligned}
\tag{8.14}
$$

For $f \in C_0^\infty(\mathbf{R})$, we write $f(t)=(t+i)^{-N}g(t)$ with $g \in C_0^\infty(\mathbf{R})$. Then

$$
\begin{aligned}
&[f(Q.)]_0^1 = [(i+Q.)^{-N}g(Q.)]_0^1 \\
&= -\tfrac{1}{\pi}\int \tfrac{\partial\widetilde{g}}{\partial\overline{z}}(z)[(i+Q.)^{-N}(z-Q.)^{-1}]_0^1 L(dz),
\end{aligned}
\tag{8.15}
$$

where $\widetilde{g} \in C_0^\infty(\mathbf{C})$ is an almost analytic extension of g. Using (8.14), we get

$$\|[f(Q.)]_0^1\|_{\mathrm{tr}} = \mathcal{O}(h^{-n}), \tag{8.16}$$

and we conclude (as already mentioned) that

$$\|[(1-\chi)f(P.)(1-\chi)]_0^1\|_{\mathrm{tr}} = \mathcal{O}(h^{-n}). \tag{8.17}$$

We have then proved everything in Proposition 8.1, except that the right hand side of (8.4) is independent of the choice of χ, which we leave to the reader. #

Remark. Let $f \in C_0^\infty(]-\infty,0[)$. Possibly after increasing R_0, we may assume that $(z-Q)^{-1}$ exists and is equal to $\mathcal{O}(1)$ for z in some fixed complex neighborhood of $\mathrm{supp}\, f$ so that $f(Q)=0$. Lemma 8.3 together with the operator Cauchy formula then shows that $(1-\chi)f(P)(1-\chi)$ is trace negligible, now in the sense that the trace class norm is $\mathcal{O}(h^\infty)$. Choose $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ with $1_{B(0,R_0)} < \chi_0 < \chi$. Then $(1-\chi_0)f(P)(1-\chi_0)$ is also trace negligible and

$$\mathrm{tr}\, f(P) = \mathrm{tr}\, \chi f(P)\chi + \mathrm{tr}\,(1-\chi)f(P)\chi + \mathrm{tr}\, \chi f(P)(1-\chi) + \mathcal{O}(h^\infty)\ .$$

Here $\mathrm{tr}\, \chi f(P)(1-\chi) = \mathrm{tr}\, \chi f(P)(1-\chi)(1-\chi_0)^2 = \mathrm{tr}\, \chi(1-\chi_0)f(P)(1-\chi_0)(1-\chi) = \mathcal{O}(h^\infty)$. Similarly $\mathrm{tr}\,(1-\chi)f(P)\chi = \mathcal{O}(h^\infty)$ so $\mathrm{tr}\, f(P) = \mathrm{tr}\, \chi f(P)\chi +$

$\mathcal{O}(h^\infty)$ Similarly $\operatorname{tr} f(P^\sharp) = \operatorname{tr} \chi f(P^\sharp)\chi + \mathcal{O}(h^\infty)$ and from Lemma ltrace1.3 and the Cauchy formula we have $tr\chi f(P)\chi = \operatorname{tr} \chi f(P^\sharp)\chi + \mathcal{O}(h^\infty)$. We conclude that

$$\operatorname{tr} f(P) = \operatorname{tr} f(P^\sharp) + \mathcal{O}(h^\infty) \ .$$

Remark. If P_0, P_1 satisfy the assumptions of Proposition ltrace1.1, then uniformly with respect to $\lambda \geq 1$, the same holds for the operators $\lambda^{-1}P_0$, $\lambda^{-1}P_1$, provided that we replace h by the new semi-classical parameter $h/\sqrt{\lambda}$. The same holds for Theorem 8.2. In particular, if P_0, P_1 are independent of h and satisfy all assumptions of Proposition ltrace1.1 (or Theorem ltrace1.2) which make sense for $h = 1$, then $\lambda^{-1}P_0$, $\lambda^{-1}P_1$ do the same with the semi-classical parameter $h = 1/\sqrt{\lambda}$.

8.2 Proof of the local trace formula.

We choose $m \in \mathbf{N}$ large enough to get nice trace class properties for some operators below. We take $\theta = \theta_0$. Let Ω_+ be the intersection of Ω with the sector $0 \leq \arg z < \epsilon_0$ (for some $\varepsilon_0 > 0$), define W_+ similarly and recall that W_-, Ω_- were defined prior to Theorem 8.2. Fix a point $z_0 \in e^{i[3\epsilon_0, \min(\pi, 2\pi - 2\theta - 3\epsilon_0)]}]0, +\infty[$ away from $\sigma(P_\cdot)$ and $\sigma(P_{\cdot,\theta})$ (where we use the observation that we can always replace ϵ_0 by some smaller number without changing anything in the conclusion of Theorem 8.2) and write

$$f(z) = (z - z_0)^{-m} g(z). \tag{8.18}$$

Then

$$(\chi f)(P_\cdot) = (P_\cdot - z_0)^{-m}(\chi g)(P_\cdot) = \tag{8.19}$$
$$-\frac{1}{\pi}\int g(z)\frac{\partial \widetilde{\chi}}{\partial \overline{z}}(z)(P_\cdot - z_0)^{-m}(z - P_\cdot)^{-1}L(dz),$$

where $\widetilde{\chi} \in C_0^\infty(\Omega)$ is an almost analytic extension of χ with support in a small neighborhood of J_+ and equal to 1 near $\overline{I}_+$. We first look at

$$I_\cdot^- := -\frac{1}{\pi}\int_{\Omega_-} g(z)\frac{\partial \widetilde{\chi}}{\partial \overline{z}}(z)(P_\cdot - z_0)^{-m}(z - P_\cdot)^{-1}L(dz). \tag{8.20}$$

Let $\widehat{\chi} \in C_0^\infty(\Omega)$ be equal to 1 near $\overline{W}_-$, equal to $\widetilde{\chi}$ near J_+ and be almost analytic near $\mathbf{R}_-$, in case Ω intersects that set. If $\Omega \cap \mathbf{R}_- = \emptyset$, we can replace

$\widetilde{\chi}$ by $\widehat{\chi}$ in the last formula. More generally, we have

$$I_\cdot^- = -(\widehat{\chi}1_{\mathbf{R}_-}f)(P_\cdot) - \frac{1}{\pi}\int_{\Omega_-} g(z)\frac{\partial\widehat{\chi}(z)}{\partial\overline{z}}(P_\cdot - z_0)^{-m}(z-P_\cdot)^{-1}L(dz). \quad (8.21)$$

The support of $\partial\widehat{\chi}(z)/\partial\overline{z}$ in Ω_- does not intersect W_-, so $|f| \le 1$ there and consequently $g = \mathcal{O}(1)$ uniformly w.r.t. h on that set. Choosing m large enough, it follows from the first remark after (8.17) and the proof of Proposition 8.1, that

$$''\mathrm{tr}\,[I_\cdot^-]_0^{1}{}'' = -[\sum_{\mu\in\sigma(P_\cdot)\cap\mathbf{R}_-\cap W} f(\mu)]_0^1 + \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}). \quad (8.22)$$

Next look at

$$I_\cdot^+ := -\frac{1}{\pi}\int_{\Omega_+} g(z)\frac{\partial\widehat{\chi}(z)}{\partial\overline{z}}(P_\cdot - z_0)^{-m}(z-P_\cdot)^{-1}L(dz).$$

Green's formula gives for every $\delta > 0$:

$$\begin{aligned} I_\cdot^+ = &-\frac{1}{\pi}\int_{\Omega_+\cap\{\mathrm{Im}\, z\le\delta\}} g(z)\frac{\partial\widehat{\chi}(z)}{\partial\overline{z}}(P_\cdot - z_0)^{-m}(z-P_\cdot)^{-1}L(dz) \quad (8.23)\\ &+\frac{1}{2\pi i}\int_{\Omega_+\cap\{\mathrm{Im}\, z=\delta\}} g(z)\widehat{\chi}(z)(P_\cdot - z_0)^{-m}(z-P_\cdot)^{-1}dz, \end{aligned}$$

where the integration contour in the last integral is oriented in the direction of decreasing $\mathrm{Re}\, z$. If $\delta > 0$ is small enough (independent of h), the integrand in the first integral in (8.23) has its support in the region where $g = \mathcal{O}(1)$ and as in the proof of Proposition 8.1, we get

$$''\mathrm{tr}\,[I_\cdot^+]_0^{1}{}'' = {}''\mathrm{tr}\,[J_\cdot]_0^{1}{}'' + \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}), \quad (8.24)$$

where $J_\cdot$ denotes the last integral in (8.23).

Committing another error $\mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2})$ in the evaluation of $''\mathrm{tr}\,[J_\cdot]_0^{1}{}''$, we may replace $J_\cdot$ by

$$J_\cdot := \frac{1}{2\pi i}\int_\gamma g(z)(P_\cdot - z_0)^{-m}(z-P_\cdot)^{-1}dz, \quad (8.25)$$

where γ is the segment from b to a, where $\mathrm{Im}\, b = \mathrm{Im}\, a = \delta$, $\mathrm{Re}\, a, \mathrm{Re}\, b \in J_+$, $\mathrm{Re}\, a < \inf I_+$, $\mathrm{Re}\, b > \sup I_+$.

Proposition 8.4 *We have*

$$''\mathrm{tr}\,[J.]_0^{1}'' = ''\mathrm{tr}\,[\widetilde{J}.]_0^{1}'', \tag{8.26}$$

where

$$\widetilde{J}. = \frac{1}{2\pi i}\int_\gamma g(z)(P_{\cdot,\theta}-z_0)^{-m}(z-P_{\cdot,\theta})^{-1}L(dz), \tag{8.27}$$

and where $P_{\cdot,\theta} = P_{\cdot|\Gamma_\theta}$ is defined as in section 3, with the parameter ϵ_0 there chosen small enough depending on δ.

Proof. Take $\chi \in C^\infty(\mathbf{R};[0,1])$ with $\chi(t)=0$ for $t\le 0$, $\chi(t)=1$ for $t\ge 1$. As in the proof of Proposition 7.5 (but now with a different χ), we put

$$f_{\theta_1,\theta_2,T}(t) = f_{\theta_1}(t) + \chi(\frac{t}{T})(f_{\theta_2}(t)-f_{\theta_1}(t)),$$

and let $\Gamma_{\theta_1,\theta_2,T}$ be the corresponding contour. As in Chapter 7 we assume that
$0\le\theta_1<\theta_2\le\theta$ with $\theta_2-\theta_1$ small enough, and we let $P_{\cdot,\theta_1,\theta_2,T}$ denote the restriction of $P.$ to $\Gamma_{\theta_1,\theta_2,T}$. The proof of Proposition 8.1 can be applied to show that

$$''\mathrm{tr}\,[(P_{\cdot,\theta_1,\theta_2,T}-z_0)^{-m}(z-P_{\cdot,\theta_1,\theta_2,T})^{-1}]_0^{1}'' \tag{8.28}$$

is $\mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2})$, depends continuously on $T>0$, is equal to

$$''\mathrm{tr}\,[(P_{\cdot,\theta_2}-z_0)^{-m}(z-P_{\cdot,\theta_2})^{-1}]_0^{1}''$$

for small T, and tends to

$$''\mathrm{tr}\,[(P_{\cdot,\theta_1}-z_0)^{-m}(z-P_{\cdot,\theta_1})^{-1}]_0^{1}'',$$

when $T\to+\infty$.

To get the proposition, it suffices to show that the expression (8.28) is independent of T. Let $\Gamma_t=\{f(t,y); y\in\omega\}$, $0\le t\le 1$, be a smooth family of m.t.r. manifolds in $\mathbf{C}^n$, where ω is open and f in C^∞, with $\det(\frac{\partial f}{\partial y})\neq 0$ everywhere.

If u is holomorphic near Γ_t for some t, let $u_t(y)=u(f(t,y))$, so that u_t is the restriction $u_{|\Gamma_t}$, expressed in the parametrization $f(t,\cdot)$. Then

$$\frac{\partial u_t(y)}{\partial t} = (\frac{\partial f}{\partial t}(t,y)\cdot\frac{\partial u}{\partial x})(f(t,y)). \tag{8.29}$$

Since Γ_t is m.t.r., there exists a complex vector field $\nu_t(y, \frac{\partial}{\partial y})$ on Γ_t (tangent to Γ_t), such that

$$\frac{\partial u_t}{\partial t} = \nu_t u_t. \tag{8.30}$$

Moreover ν_t depends smoothly on (t, y) and vanishes outside the support of $\frac{\partial f}{\partial t}(t, \cdot)$.

Let P be a differential operator with holomorphic coefficients, defined near Γ_t. Define $P_t = P_{\Gamma_t}$ as in section 3, so that

$$P_t u_t = (Pu)_t, \tag{8.31}$$

when u is holomorphic near Γ_t. Then

$$\nu_t(P_t u_t) = \partial_t(P_t u_t) = (\partial_t P_t)(u_t) + P_t \partial_t u_t = (\partial_t P_t)(u_t) + P_t \nu_t u_t,$$

and since the functions of the type u_t are "sufficiently dense", we conclude that in the sense of differential operators,

$$\partial_t P_t = [\nu_t, P_t]. \tag{8.32}$$

This applies to $\Gamma_{\theta_1,\theta_2,T}$ and passes to the resolvents and their compositions:

$$\begin{aligned} &\partial_T((P_{\cdot,\theta_1,\theta_2,T} - z_0)^{-m}(z - P_{\cdot,\theta_1,\theta_2,T})^{-1}) = \\ &[\nu, (P_{\cdot,\theta_1,\theta_2,T} - z_0)^{-m}(z - P_{\cdot,\theta_1,\theta_2,T})^{-1}], \end{aligned} \tag{8.33}$$

where $\nu = \nu_{\theta_1,\theta_2,T}$ is a smooth complex vector field whose support is compact and disjoint from $\overline{B(0, R_0)}$.

When considering (8.28), we may choose the cutoff χ in (8.4) with support disjoint from that of ν, so with ψ equal to χ or to $1 - \chi$,

$$\begin{aligned} &\operatorname{tr} \chi[\nu, (P_{\cdot,\theta_1,\theta_2,T} - z_0)^{-m}(z - P_{\cdot,\theta_1,\theta_2,T})^{-1}]\psi = \\ &\operatorname{tr} \chi\nu \circ (...)^{-m}(..)^{-1}\psi - \operatorname{tr} \chi(...)^{-m}(..)^{-1} \circ \nu\psi \\ &= 0 - \operatorname{tr} (...)^{-m}(..)^{-1}\nu \circ \psi\chi = 0, \end{aligned}$$

where we used the cyclicity of the trace. The T-derivative of the expression (8.28) then reduces to the following expression, where we drop θ_1, θ_2, T for simplicity:

$$\begin{aligned}&\operatorname{tr}([(1-\chi)[\nu,(P_{\cdot}-z_0)^{-m}(z-P_{\cdot})^{-1}](1-\chi)]_0^1)\\&=\operatorname{tr}[\nu,[(1-\chi)(P_{\cdot}-z_0)^{-m}(z-P_{\cdot})^{-1}(1-\chi)]_0^1].\end{aligned}\tag{8.34}$$

Here $[(1-\chi)(P_{\cdot}-z_0)^{-m}(z-P_{\cdot})^{-1}(1-\chi)]_0^1$ is of trace class also as an operator $L^2\to H^1$, so the expression (8.34) vanishes by the cyclicity of the trace. It follows that the T-derivative of (8.28) is zero and the proof is complete. #

It remains to study "$\operatorname{tr}[\widetilde{J}_{\cdot}]_0^1$". We shall use a finite rank perturbation of $P_{\cdot,\theta}$. Let F be a smooth mapping from a neighborhood of $e^{i[-2(\theta+\epsilon_0),\epsilon_0]}[0,\infty[$ into itself such that

$$F(z)=z,\ \text{for } |z| \text{ large and for } z \text{ near } e^{-2i\theta}[0,\infty[,\tag{8.35}$$

$$\overline{\Omega} \text{ is disjoint from the image of } F.\tag{8.36}$$

If $\theta<\frac{\pi}{2}$, we further extend F to be a smooth map from a neighborhood of $e^{i[-\pi-\epsilon_0,\epsilon_0]}[0,\infty[$ to itself, by putting $F(z)=z$ for z near $e^{i[-\pi-\epsilon_0,-2\theta]}[0,\infty[$.

In the appendix (with a suitable choice of Γ_θ) we shall construct an operator $\widehat{P}_{\cdot,\theta}:\mathcal{D}_{\cdot}\to\mathcal{H}_{\cdot}$ with the following properties:

$$\begin{aligned}&K_{\cdot}:=\widehat{P}_{\cdot,\theta}-P_{\cdot,\theta} \text{ is of rank } \mathcal{O}(1)\Phi_{\max}(\tfrac{1}{h^2}) \text{ and}\\&\mathcal{O}(1):\mathcal{D}(P_{\cdot}^N)\to\mathcal{D}(P_{\cdot}^M) \text{ for all } N,M\in\mathbf{N}.\end{aligned}\tag{8.37}$$

Notice here that $\mathcal{D}(P_{\cdot}^0)=\mathcal{H}_{\cdot}$.

$$\begin{aligned}&K_{\cdot} \text{ is compactly supported in the sense that } K_{\cdot}=\chi K_{\cdot}\chi\\&\text{if } \chi\in C_0^\infty \text{ is equal to 1 on } B(0,R) \text{ for some sufficiently large } R.\end{aligned}\tag{8.38}$$

$$\begin{aligned}\text{For every } N\in\mathbf{N}, (\widehat{P}_{\cdot,\theta}-z)^{-1}&=\mathcal{O}(1):\mathcal{D}(P_{\cdot}^N)\to\mathcal{D}(P_{\cdot}^{N+1}),\\&\quad\text{uniformly for } z\in\overline{\Omega}.\end{aligned}\tag{8.39}$$

Since we shall always work with operators on Γ_θ in the remainder of the proof, we drop the subscript θ most of the time and write $P_{\cdot}$ instead of $P_{\cdot,\theta}$.

Write

$$z - P_{\cdot} = (1 + \widetilde{K}_{\cdot})(z - \widehat{P}_{\cdot}), \text{ where } \widetilde{K}_{\cdot} = K_{\cdot}(z - \widehat{P}_{\cdot})^{-1}, \tag{8.40}$$

so that

$$(z - P_{\cdot})^{-1} = (z - \widehat{P}_{\cdot})^{-1}(1 + \widetilde{K}_{\cdot})^{-1}. \tag{8.41}$$

Using the resolvent identity,

$$(z - P_{\cdot})^{-1} - (z - \widehat{P}_{\cdot})^{-1} = -(z - P_{\cdot})^{-1} K_{\cdot}(z - \widehat{P}_{\cdot})^{-1},$$

we can decompose the right hand side of (8.26) as $\mathrm{I} + \mathrm{II}$, where

$$\mathrm{I} = {}''\mathrm{tr}\,[\frac{1}{2\pi i}\int_\gamma g(z)(P_{\cdot} - z_0)^{-m}(z - \widehat{P}_{\cdot})^{-1}dz]_0^1{}'', \tag{8.42}$$

$$\mathrm{II} = -{}''\mathrm{tr}\,[\frac{1}{2\pi i}\int_\gamma g(z)(P_{\cdot} - z_0)^{-m}(z - P_{\cdot})^{-1}K_{\cdot}(z - \widehat{P}_{\cdot})^{-1}dz]_0^1{}''. \tag{8.43}$$

Let $\widetilde{\gamma}$ be a smooth contour from b to a (as was the case with γ) and contained in $\Omega \setminus W$. The integrand in I is holomorphic in Ω, so we may replace γ by $\widetilde{\gamma}$ in (8.42). Along $\widetilde{\gamma}$, we have $g = \mathcal{O}(1)$ (uniformly w.r.t. h) and if we choose a cutoff χ in Lemma 8.3 equal to 1 on a sufficiently large bounded (sub)set (of Γ_θ), we can apply the proof of Proposition 8.1, to obtain:

$$''\mathrm{tr}\,[(P_{\cdot} - z_0)^{-m}(z - \widehat{P}_{\cdot})^{-1}]_0^1{}'' = \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}). \tag{8.44}$$

We therefore get

$$\mathrm{I} = \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}). \tag{8.45}$$

Since $K_{\cdot}$ are of finite rank, we can write

$$\mathrm{II} = -[\mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma g(z)(P_{\cdot} - z_0)^{-m}(z - P_{\cdot})^{-1}K_{\cdot}(z - \widehat{P})^{-1}dz]_0^1. \tag{8.46}$$

Here we can replace $(P_{\cdot} - z_0)^{-m}$ by $(z - z_0)^{-m}$ because the difference has an integrand which is holomorphic in Ω, and the contour γ can then be replaced by $\widetilde{\gamma}$. It follows that the difference gives a contribution $\mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2})$ to II. Recalling (8.18), we get:

$$\mathrm{II} = -[\frac{1}{2\pi i}\int_\gamma f(z)\mathrm{tr}\,((z - P_{\cdot})^{-1}K_{\cdot}(z - \widehat{P}_{\cdot})^{-1})dz]_0^1 + \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}). \tag{8.47}$$

Now use (8.41) and the cyclicity of the trace:

$$-\mathrm{tr}\,((z-P.)^{-1}K.(z-\widehat{P}.)^{-1}) = -\mathrm{tr}\,((z-\widehat{P}.)^{-1}(1+\widetilde{K}.)^{-1}K.(z-\widehat{P}.)^{-1})$$

$$= -\mathrm{tr}\,((1+\widetilde{K}.)^{-1}K.(z-\widehat{P}.)^{-2}) = \mathrm{tr}\,((1+\widetilde{K}.(z))^{-1}\tfrac{\partial}{\partial z}\widetilde{K}.(z)). \tag{8.48}$$

Remark. In the last equality in (8.48) we used that $K.$ is independent of z. It may be of interest to see what would happen if $K. = K.(z)$ (and $\widehat{P}. = \widehat{P}.(z)$) depends holomorphically on z. Differentiating the first identity in (8.40), we get

$$-\widetilde{K}.(z-\widehat{P}.)^{-1} = \frac{\partial \widetilde{K}.}{\partial z} - (1+\widetilde{K}.)\frac{\partial \widehat{P}.}{\partial z}(z-\widehat{P}.)^{-1}. \tag{8.49}$$

Using this in the third expression in (8.48), we get

$$\begin{aligned} &-\mathrm{tr}\,((z-P.)^{-1}K.(z-\widehat{P}.)^{-1}) = \\ &\mathrm{tr}\,((1+\widetilde{K}.)^{-1}\frac{\partial \widetilde{K}.}{\partial z}) - \mathrm{tr}\,((1+\widetilde{K}.)^{-1}(1+\widetilde{K}.)\frac{\partial \widehat{P}.}{\partial z}(z-\widehat{P}.)^{-1}) \\ &= \mathrm{tr}\,((1+\widetilde{K}.)^{-1}\frac{\partial \widetilde{K}.}{\partial z}) - \mathrm{tr}(\frac{\partial \widehat{P}.}{\partial z}(z-\widehat{P}.)^{-1}). \end{aligned} \tag{8.50}$$

Here the last term is holomorphic in Ω and we get (8.51) below also in this more general case.

Substitution of (8.48) into (8.47) gives

$$\mathrm{II} = [\frac{1}{2\pi i}\int_\gamma f(z)\mathrm{tr}\,((1+\widetilde{K}.(z))^{-1}\frac{\partial \widetilde{K}.}{\partial z})dz]_0^1 + \mathcal{O}(1)\Phi_{\mathrm{max}}(\frac{1}{h^2}). \tag{8.51}$$

With (8.40) in mind we consider an identity of the form

$$A(z) = B(z)C(z), \tag{8.52}$$

where $C(z) : \mathcal{H}_1 \to \mathcal{H}_2$, $B(z) : \mathcal{H}_2 \to \mathcal{H}_3$ depend holomorphically on z in some domain, and $\partial_z B(z)$ is of trace class. Restricting the attention to some subdomain, where B and C have bounded inverses, we get after differentiating the equation (8.52) and using the cyclicity of the trace:

$$\mathrm{tr}\,(B^{-1}\partial_z B) = \mathrm{tr}\,(A^{-1}\partial_z A - C^{-1}\partial_z C). \tag{8.53}$$

Let Γ be a relatively compact subdomain with smooth boundary and assume that C^{-1} exists everywhere on the closure of Γ. Also assume that $B(z)^{-1}$ exists everywhere on the boundary of Γ. Then for f holomorphic in Γ and continuous up the boundary, we get

$$\begin{aligned} &\operatorname{tr}\Big(\frac{1}{2\pi i}\int_{\partial\Gamma} f(z)B(z)^{-1}\partial B(z)dz\Big) = \\ &\operatorname{tr}\frac{1}{2\pi i}\int_{\partial\Gamma} f(z)(A^{-1}\partial_z A - C^{-1}\partial_z C)dz = \operatorname{tr}\Big(\frac{1}{2\pi i}\int_{\partial\Gamma} f(z)A^{-1}\partial_z A dz\Big). \end{aligned} \tag{8.54}$$

We apply this to (8.40) with $\Gamma \subset\subset \Omega$ and $\partial\Gamma$ avoiding the resonances, and get:

$$\operatorname{tr}\frac{1}{2\pi i}\int_{\partial\Gamma} f(z)(1+\widetilde{K}.(z))^{-1}\partial_z\widetilde{K}.dz = \sum_{\lambda\in\operatorname{Res}(P.)\cap\Gamma} f(\lambda). \tag{8.55}$$

Notice that we could have taken the trace inside the integral, that $\operatorname{tr}((1+\widetilde{K}.))^{-1}\partial_z\widetilde{K}.) = \partial_z \log\det(1+\widetilde{K}.(z))$ and that the resonances are precisely the zeros of $\det(1+\widetilde{K}.(z))$ and that the multiplicities agree.

The remainder of the proof will have common features with a proof of Lidskii's theorem, close to the one in [31], of which we gave a variant in Chapter 5. Put

$$D.(z;h) := \det(1+\widetilde{K}.) = \mathcal{O}(1)e^{\mathcal{O}(1)\Phi.(h^{-2})}. \tag{8.56}$$

Since we will work separately with the two operators $P.$, we drop the subscript $\cdot$ most of the time in the following. In $\Omega_{+,\delta} := \{z\in\Omega_+; \operatorname{Im} z > \delta\}$ with $\delta > 0$ (suitably chosen and independent of h), not only $(z-\widehat{P}) : \mathcal{D}\to\mathcal{H}$ is uniformly invertible, but also $(z-P)$, and hence

$$(1+\widetilde{K})^{-1} = (z-\widehat{P})(z-P)^{-1} = \mathcal{O}(1) : \mathcal{H}\to\mathcal{H}.$$

It follows that $(1+\widetilde{K})^{-1} = 1 + (P-\widehat{P})(z-P)^{-1}$ with

$$\|(P-\widehat{P})(z-P)^{-1}\|_{\mathrm{tr}} = \mathcal{O}(1)\Phi(\frac{1}{h^2}),$$

giving an upper bound for the determinant of $(1+\widetilde{K})^{-1}$ as in (8.56):

$$|D(z;h)| \ge e^{-C\Phi(h^{-2})},\ z\in\Omega_{+,\delta}. \tag{8.57}$$

Let $N = N(P,\Omega;h)$ be the number of resonances in Ω counted with their multiplicity. (8.56) remains valid in a slightly larger domain and combining this with (8.57) and Jensen's inequality, we get

$$N(P,\Omega;h) \leq C\Phi(\frac{1}{h^2}). \tag{8.58}$$

At this point, we could apply an estimate of H. Cartan about lower bounds of holomorphic functions outside small unions of circles around the zeros as in [81], but we prefer to keep the argument of [75, 76], which is simple, more or less standard and gives precisely what is needed.

Let z_j, $j = 1,..,N$ be the resonances in Ω repeated according to their multiplicity and put

$$D_w(z;h) = \prod_{j=1}^{N}(z - z_j). \tag{8.59}$$

Thanks to (8.58), we have

$$|D_w(z;h)| \leq e^{C\Phi(h^{-2})} \text{ in } \Omega, |D_w(z;h)| \geq e^{-C\Phi(h^{-2})} \text{ in } \Omega_{+,\delta}. \tag{8.60}$$

In $\Omega \setminus \Omega_{+,\delta}$ we can get the same lower bound, if we avoid to go too close to the z_j. For that, we first establish the simple

Lemma 8.5 *Let $x_1,..,x_N \in \mathbf{R}$ and let $I \subset \mathbf{R}$ be an interval of length $|I| \in]0,+\infty[$. Then there exists $x \in I$, such that*

$$\prod_{j=1}^{N} |x - x_j| \geq e^{-N(1+\log\frac{2}{|I|})}.$$

Proof. Consider $F(x) = \sum_{j=1}^{N} \log \frac{1}{|x-x_j|}$. We have

$$\int_I \log \frac{1}{|x - x_j|} dx \leq 2\int_0^{|I|/2} \log \frac{1}{t} dt = |I|(1 + \log \frac{2}{|I|}),$$

since the first integral takes its largest possible value when x_j is the midpoint of I. It follows that

$$\int_I F(x)dx \leq N|I|(1 + \log \frac{2}{|I|}).$$

We can therefore find $x \in I$, such that $F(x) \leq N \log(1+\log \frac{2}{|I|})$, i.e.

$$\prod_{j=1}^{N} |x-x_j| = e^{-F(x)} \geq e^{-N(1+\log \frac{2}{|I|})}.$$

#

Let $\widetilde{\gamma} = \widetilde{\gamma}_\alpha$, $\alpha \in J$ be a smooth family of smooth curves joining b to a, where J is a bounded interval of length $\neq 0$. We let $\widetilde{\gamma}_\alpha$ move transversally in $\Omega_{+,\delta/2}$ when α varies in such a way that for every $z \in \Omega_-$ we have:
1) If z belongs to $\widetilde{\gamma}_\alpha$ for some α, then this α is unique, $\alpha = \alpha(z)$ and $\mathrm{dist}\,(z, \widetilde{\gamma}_\beta) \geq |\beta - \alpha(h)|$ for all $\beta \in J$.
2) If z belongs to no $\widetilde{\gamma}_\beta$, then $\mathrm{dist}\,(z, \widetilde{\gamma}_\beta) \geq \mathrm{dist}\,(\beta, \partial J)$ for every $\beta \in J$.

It follows from Lemma 8.5, that if $I \subset J$ is an interval of length > 0, then we can find $\alpha \in I$ such that

$$|D_w(z;h)| \geq e^{-C\Phi(h^{-2})},\ z \in \widetilde{\gamma}_\alpha, \tag{8.61}$$

where $C = C_{|I|}$. We factorize D:

$$D(z;h) = G(z;h) D_w(z;h)\ z \in \Omega, \tag{8.62}$$

where G and $1/G$ are holomorphic in Ω. Combining (8.60), (8.61), (8.62), we get

$$|G(z;h)| \leq e^{C\Phi(h^{-2})},\ z \in \Omega_{+,\delta} \cup \widetilde{\gamma}_\alpha. \tag{8.63}$$

The maximum principle gives

$$|G(z;h)| \leq e^{C\Phi(h^{-2})},\ z \in \widetilde{\Omega}, \tag{8.64}$$

where $\widetilde{\Omega} \subset\subset \Omega$ is any simply connected relatively open h-independent set with $\overline{W}$ in its (relative) interior. In fact, it suffices to choose the family $\widetilde{\gamma}_\alpha$ with $\widetilde{\gamma}_\alpha \cap \widetilde{\Omega} \cap \Omega_- = \emptyset$.

The relations (8.60, (8.57) imply that

$$|G(z;h)| \geq e^{-C\Phi(h^{-2})},\ z \in \Omega_{+,\delta}. \tag{8.65}$$

Choose $C > 0$ large enough, so that

$$\ell(z;h) := C\Phi(\frac{1}{h^2}) - \log|G(z;h)| \geq 0. \tag{8.66}$$

Notice that ℓ is harmonic. Harnack's inequality tells us that for every $K \subset\subset \widetilde{\Omega}$, there is a constant $C_K \geq 1$ such that for every non-negative harmonic function $\tilde{\ell}$ on $\tilde{\Omega}$:

$$\sup_K \tilde{\ell} \leq C_K \inf_K \tilde{\ell}, \tag{8.67}$$

This applies to ℓ and after an arbitrarily small decrease of $\widetilde{\Omega}$, we have (8.67) with $K = \widetilde{\Omega}, \tilde{\ell} = \ell$ and if we use (8.65), we get $\ell(z;h) \leq C\Phi(\frac{1}{h^2})$ on $\Omega_{+,\delta}$ and hence by (8.67) (with $K = \widetilde{\Omega}$):

$$\ell(z;h) \leq C\Phi(\frac{1}{h^2}) \text{ on } \widetilde{\Omega}. \tag{8.68}$$

We conclude that $\log |G| \geq -C\Phi(\frac{1}{h^2})$ on $\widetilde{\Omega}$ and with (8.64) we get

$$|\log |G(z;h)|| \leq C\Phi(\frac{1}{h^2}),\ z \in \widetilde{\Omega}. \tag{8.69}$$

Since $\log |G(z;h)| = \mathrm{Re} \log G$ is harmonic, we get after an arbitrarily small decrease of $\widetilde{\Omega}$:

$$\nabla(\mathrm{Re}\ \log G) = \mathcal{O}(1)\Phi(\frac{1}{h^2}),\ z \in \widetilde{\Omega}. \tag{8.70}$$

The Cauchy Riemann equations for $\log G$ imply

$$\frac{d}{dz} \log G = \mathcal{O}(1)\Phi(\frac{1}{h^2}),\ z \in \widetilde{\Omega}. \tag{8.71}$$

Let $\widetilde{\gamma}$ be a curve from b to a in $\widetilde{\Omega} \setminus W$ and which avoids the resonances. From (8.58), (8.59) we get

$$\int_{\widetilde{\gamma}} f(z;h) \frac{d}{dz} \log D_w(z;h) dz = \mathcal{O}(1)\Phi(\frac{1}{h^2}). \tag{8.72}$$

Since

$$\frac{d}{dz} \log D = \frac{d}{dz} \log G + \frac{d}{dz} \log D_w,$$

we get from (8.71), (8.72):

$$\int_{\widetilde{\gamma}} f(z;h) \frac{d}{dz} \log D\, dz = \mathcal{O}(1)\Phi(\frac{1}{h^2}). \tag{8.73}$$

We finally return to (8.51), where we recall (8.55) and the subsequent identity. Together with (8.73), this gives:

$$\begin{aligned}
&\mathrm{II} = \qquad\qquad (8.74)\\
&\Big[\sum_{\substack{\mathrm{Res}(P.)\ni z\\ \text{between } \widetilde{\gamma} \text{ and } \gamma}} f(z;h) + \frac{1}{2\pi i}\int_{\widetilde{\gamma}} f(z;h)\frac{d}{dz}(\log D.(z))dz\Big]_0^1 + \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2})\\
&= \Big[\sum_{\substack{\mathrm{Res}(P.)\ni z\\ \text{between } \widetilde{\gamma} \text{ and } \gamma}} f(z;h)\Big]_0^1 + \mathcal{O}(1)\Phi_{\max}(\frac{1}{h^2}).
\end{aligned}$$

Here we reintroduce the subscripts "." and recall that $\Phi_{\max} = \max(\Phi_0, \Phi_1)$. Since the total number of z_j's is $\mathcal{O}(1)\Phi(\frac{1}{h^2})$ and $|f| \le 1$ for $z \in \Omega \setminus W$, we may replace the last sum in (8.74) by

$$\sum_{z\in W_-\cap \mathrm{Res}(P.)} f(z;h)$$

and the proof of Theorem 8.2 is complete. #

Remark. Applying Lemma 8.3 or rather the equivalent estimate on $\int_I F(x)dx$ in its proof, we see that if we take a family of contours $\widetilde{\gamma}_\alpha$, $\alpha \in J$, $|J| > 0$, as earlier in the proof, then (for every h) we can find $\alpha \in J$, such that (8.73) strengthens to

$$\int_{\widetilde{\gamma}_\alpha} |\frac{d}{dz}\log D||dz| = \mathcal{O}(1)\Phi(\frac{1}{h^2}). \qquad (8.75)$$

In fact, in view of (8.71) it suffices to establish this with D replaced by D_w.

Appendix: construction of $\widehat{P}_{\cdot,\theta}$.

We shall construct an operator $\widehat{P}_{\cdot,\theta}$, which satisfies (8.37)–(8.39) and for that we shall use the function F which was introduced prior to those equations. Since we shall work with the two operators $P.$ separately we drop the subscript $\cdot$. Let p_θ be the semi-classical principal symbol of P_θ, defined as $\sum_{|\alpha|\le 2} b_\alpha(x;h)\xi^\alpha \mod \mathcal{O}(h\langle\xi\rangle/\langle x\rangle)$, where $P_\theta = \sum_{|\alpha|\le 2} b_\alpha(x;h)(hD_x)^\alpha$. Choosing Γ_θ suitably (more precisely with R_1 sufficiently large and ϵ_0 sufficiently small in the construction of f_θ in section 3), we see that $F \circ p_\theta$ is a well-defined symbol with values away from $\overline{\Omega}$, such that $F \circ p_\theta = p_\theta$, when $|x| \ge R_2$, $R_2 > R_1$. Here we recall that Γ_θ coincides with $\mathbf{R}^n$ for $|x| \le R_1$.

Let $P^\sharp$ be a reference operator as in section 2 and assume that the corresponding torus T is large enough so that $B(0,R_1)$ is naturally contained in T. If $f=F_{|\mathbf{R}}$, then

$$f(P^\sharp)=P^\sharp+g(P^\sharp)=P^\sharp+K^\sharp, \tag{A.1}$$

where $f(E)=E+g(E)$, $g\in C_0^\infty(\mathbf{R})$, and

$$K^\sharp=\mathcal{O}(1):\mathcal{H}^\sharp\to\mathcal{H}^\sharp,\ \operatorname{rank}K^\sharp=\mathcal{O}(1)\Phi(\frac{1}{h^2}). \tag{A.2}$$

More generally $K^\sharp$ is uniformly $\mathcal{O}(1)$ as a bounded operator from the domain of any power of $P^\sharp$ to the domain of any other power of $P^\sharp$. Moreover

$$(z-f(P^\sharp))^{-1}=\mathcal{O}(1), \tag{A.3}$$

uniformly for $z\in\Omega$ and in h. Let $Q^\sharp$ be a self-adjoint differential operator as in the construction of $P^\sharp$, so that $Q^\sharp=P^\sharp$ in $T\setminus B(0,R_0)$. As in the proof of Lemma 8.3, we see that if $\chi\in C_0^\infty(T\setminus\overline{B(0,R_0)})$, then

$$\chi(z-P^\sharp)^{-1}\chi-\chi(z-Q^\sharp)^{-1}\chi \text{ is } \mathcal{O}(\frac{h^N}{|\operatorname{Im}z|^{M(N)}}):H^{-N}\to H^N,\ \forall N\in\mathbf{N}. \tag{A.4}$$

Using the Cauchy formula (7.21) for $g(P^\sharp)$, $g(Q^\sharp)$, we see that

$$\chi f(P^\sharp)\chi-\chi f(Q^\sharp)\chi=\mathcal{O}(h^N):H^{-N}\to H^N,\forall N\in\mathbf{N} \tag{A.5}$$

for χ as in (A.4).

Using a result of Helffer-Robert, which can also be proved with the Cauchy formula (7.21) (see [21] and further references there), we know that $\chi f(Q^\sharp)\chi$ is a h-pseudo-differential operator on T of the natural class, with leading symbol $\chi(x)^2f(p^\sharp(x,\xi))$, where $p^\sharp$ is the semiclassical leading symbol of $Q^\sharp$.

Let $1=\chi_0+\chi_1+\chi_2$, where $\chi_0\in C_0^\infty(B(0,R_1);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$, $\chi_1\in C_0^\infty(\Gamma_\theta;[0,1])$, $\chi_0+\chi_1=1$ near $\overline{B(0,R_2)}$. Then $\chi_2\in C_b^\infty(\Gamma_\theta;[0,1])$ has its support disjoint from $\overline{B(0,R_2)}$. Let $\chi_j\prec\widetilde{\chi}_j$, where $\widetilde{\chi}_j\in C^\infty(\Gamma_\theta;[0,1])$ has its support close to that of χ_j, and define

$$\widetilde{P}_\theta:=\widetilde{\chi}_0f(P^\sharp)\chi_0+\widetilde{\chi}_1R_F\chi_1+\widetilde{\chi}_2P_\theta\chi_2, \tag{A.6}$$

where R_F is an h-pseudodifferential operator with leading symbol $F(p_\theta)$ such that the total (Weyl) symbol (with Γ_θ identified with $\mathbf{R}^n$) of R_F-P_θ (for x near $\operatorname{supp}\chi_1$) has compact support in ξ.

Lemma 8.6 *If $h > 0$ is small enough, then $z - \widetilde{P}_\theta$ is invertible for every $z \in \Omega$ and, for every $N \in \mathbf{N}$, the inverse satisfies*

$$(z - \widetilde{P}_\theta)^{-1} = \mathcal{O}_N(1) : \mathcal{D}^N \to \mathcal{D}^{N+1}, \tag{A.7}$$

uniformly for $z \in \Omega$, where $\mathcal{D}^N$ is the domain of P^N, so that $\mathcal{D}^0 = \mathcal{H}$.

Proof. The operator $z - \widetilde{P}_\theta : \mathcal{D} \to \mathcal{H}$ is Fredholm of index 0, for $z \in \Omega$, so to get invertibility and the estimate (A.7) in the case $N = 0$, it is enough to show the a priori estimate:

$$\|u\|^2 \leq C\|(z - \widetilde{P}_\theta)u\|^2, \tag{A.8}$$

for $u \in \mathcal{D}$. Let $\psi_0, \psi_1, \psi_2 \in C_b^\infty(\mathbf{C}_\theta; [0,1])$ have the same support properties as χ_0, χ_1, χ_2 and with

$$1 = \psi_0^2 + \psi_1^2 + \psi_2^2. \tag{A.9}$$

Assume that $\psi_0 \prec \chi_0$. Then for some $r_1 > 0$ and for all $z \in \Omega$

$$\|(z - \widetilde{P}_\theta)\psi_0 u\|^2 = \|(z - f(P^\sharp))\psi_0 u\|^2 + \mathcal{O}(h^\infty)\|u\|^2,$$

and for $j = 1, 2$:

$$\|(z - \widetilde{P}_\theta)\psi_j u\|^2 \geq r_1^2 \|\psi_j u\|^2.$$

Then

$$\begin{aligned}\|(z - \widetilde{P}_\theta)u\|^2 = \sum_0^2 \|\psi_j (z - \widetilde{P}_\theta)u\|^2 &\geq \sum_0^2 (\|(z - \widetilde{P}_\theta)\psi_j u\| - \|[\psi_j, \widetilde{P}_\theta]u\|)^2 \\ &\geq r_1^2\|u\|^2 - \mathcal{O}(h)\|u\|_{\mathcal{D}}^2.\end{aligned}$$

But

$$\|u\|_{\mathcal{D}_\theta} \leq \mathcal{O}(1)(\|(z - \widetilde{P}_\theta)u\| + \|u\|),$$

so we get (A.8) and the stronger a priori estimate

$$\|u\|_{\mathcal{D}} \leq C\|(z - \widetilde{P}_\theta)u\|, \tag{A.10}$$

which is precisely (A.7) for $N = 0$.

If $(z - \widetilde{P}_\theta)u = v$, with $u, v \in \mathcal{D}$, then

$$(z_0 - P_\theta)u = v + ((z_0 - z) + (\widetilde{P}_\theta - P_\theta))u \in \mathcal{D}, \tag{A.11}$$

and applying $(z-\widetilde{P}_\theta)$ and using (A.10), we see that $\|(z_0-P_\theta)u\|_{\mathcal{D}} \le C\|v\|_{\mathcal{D}}$. This gives the case $N=1$ of (A.7) and we can continue this argument by iteration. #

Write

$$\widetilde{P}_\theta = P_\theta + \widetilde{\chi}_0 g(P^\sharp)\chi_0 + \widetilde{\chi}_1(R_F - P_\theta)\chi_1.$$

We can find T_F of finite rank $\mathcal{O}(h^{-n})$ and $=\mathcal{O}(1): H^{s_1}\to H^{s_2}$ for all $s_1, s_2 \in \mathbf{R}$ such that

$$\widetilde{\chi}_1((R_F-P_\theta)-T_F)\chi_1 = \mathcal{O}(h^\infty): H^{s_1}\to H^{s_2}$$

for all $s_1, s_2\in\mathbf{R}$. Put

$$\widehat{P}_\theta = P_\theta + \widetilde{\chi_0} g(P^\sharp)\chi_0 + \widetilde{\chi}_1 T_F\chi_1. \tag{A.12}$$

Then it easy to see that (8.37)-(8.39) hold.

9 Clouds of resonances for $-h^2\Delta + V(x)$

The applications that we develop in this chapter exploit very much the semi-classical nature of the operator. In chapter 10 we shall get other applications, also for operators that are not semi-classical to start with, and which exploit the existence of closed classical trajectories. The proof of the results in this chapter will also use a semi-classical trace formula of D. Robert, and we start by discussing certain distributions on the real axis that appear in connection with that formula.

Let V_0, V_1 be continuous real-valued functions on $\mathbf{R}^n$ which tend to 0, when $|x|\to\infty$, and satisfy

$$|V_1(x)-V_0(x)| \le C\langle x\rangle^{-\widetilde{n}}, \tag{9.1}$$

where $\widetilde{n} > n$.

For $E>0$, let $\nu_{+,\cdot}(E) = \int_{V_\cdot(x)\ge E} dx$. This is a decreasing function of E, so $\mu_{+,\cdot}(E) := -\frac{d}{dE}\nu_{+,\cdot}(E)$ is a positive measure on $]0,+\infty[$ with support equal to $]0,\sup V_\cdot]$. Similarly, for $E<0$, we put $\nu_{-,\cdot}(E) = \int_{V_\cdot(x)\le E} dx$ (an increasing function of E,) $\mu_{-,\cdot}(E) = \frac{d}{dE}\nu_{-,\cdot}(E)$ which is a positive measure on $]-\infty, 0[$ with support in $[\inf V_\cdot, 0[$.

For $\phi \in C_0^\infty(\mathbf{R})$, we put

$$\langle \mu, \phi \rangle = \int [\phi \circ V_\cdot]_0^1 dx = \int (\phi(V_1(x)) - \phi(V_0(x))) dx$$

Clearly μ is a distribution of order ≤ 1 with support contained in $[\min \inf V_\cdot, \max \sup V_\cdot]$, $\int \mu(E) dE = 0$, and

$$\mu_{|\mathbf{R}_\pm} = [\mu_{\pm,\cdot}]_0^1. \tag{9.2}$$

For $f \in C_0^\infty(\mathbf{R})$, we put

$$\langle \omega, f \rangle = \iint [f(\xi^2 + V_\cdot(x))]_0^1 dx d\xi,$$

so that ω is a distribution on $\mathbf{R}$ of order ≤ 1 with

$$\operatorname{supp} \omega \subset [\min \inf V_\cdot, +\infty[.$$

We have with $E_+ := \max(E, 0)$ and $*$ denoting convolution:

$$\begin{aligned}
\int f(\xi^2 + V_\cdot(x)) d\xi &= \int_0^\infty f(E + V_\cdot(x)) d(\operatorname{Vol}\{\xi \in \mathbf{R}^n; |\xi| \leq \sqrt{E}\}) \\
&= \operatorname{Vol}(B_{\mathbf{R}^n}(0,1)) \int f(V_\cdot(x) + E) d(E_+^{n/2}) \\
&= \frac{n}{2} \operatorname{Vol}(B_{\mathbf{R}^n}(0,1)) \int f(V_\cdot(x) + E) E_+^{\frac{n}{2}-1} dE \\
&= C_n (f * (-\cdot)_+^{\frac{n}{2}-1})(V_\cdot(x)),
\end{aligned}$$

where $C_n > 0$. Then

$$\langle \omega, f \rangle = C_n \langle \mu, f * (-\cdot)_+^{\frac{n}{2}-1} \rangle = C_n \langle \mu * (\cdot)_+^{\frac{n}{2}-1}, f \rangle,$$

so

$$\omega = C_n \mu * (\cdot)_+^{\frac{n}{2}-1}. \tag{9.3}$$

In general, if $\alpha > -1$ and if $H_\alpha(x) = x_+^\alpha$, then $\widehat{H}_\alpha(\xi) = C_\alpha(\xi - i0)^{-1-\alpha}$, where $C_\alpha \neq 0$. Let $E_\alpha \in \mathcal{S}'(\mathbf{R})$ be the inverse Fourier transform of $C_\alpha^{-1}(\xi - i0)^{1+\alpha}$. Then $\operatorname{supp} E_\alpha \subset [0, +\infty[$, $H_\alpha * E_\alpha = \delta$. When $1 + \alpha \in \mathbf{N}$, then E_α is a constant times some derivative of the delta function, and in general, we have

$$E_{\alpha_{|]0,+\infty[}}(x) = \widetilde{C}_\alpha x^{-2-\alpha}.$$

We can now invert (9.3) and get

$$\mu = \frac{1}{C_n} E_{\frac{n}{2}-1} * \omega. \tag{9.4}$$

If near some point $t_0 \in \mathbf{R}$, we know that μ extends to a holomorphic function, then by contour deformation in (9.3), we see that ω will have the same property. (It is easy to see how to get such a bounded extension by contour deformation, and an easy way to see that this extension is also holomorphic, is to regularize ω and $E_{\frac{n}{2}-1}$ by convolving with Gaussians, and then let the Gaussians converge to the delta-function.) The converse implication holds by (9.4). It follows that μ and ω have the same analytic singular support. Since μ is a real distribution we know that its analytic wavefront set $\mathrm{WF}_a(\mu)$ is of the form $\{(x,\xi) \in \mathbf{R} \times (\mathbf{R} \setminus \{0\}); x \in \mathrm{sing\,supp}_a(\mu)\}$. The same holds for ω and since these two distributions have the same analytic singular support, we conclude that

$$\mathrm{WF}_a(\mu) = \mathrm{WF}_a(\omega). \tag{9.5}$$

We refer to [73] for basic properties of the analytic wavefront set WF_a.

We now discuss resonances close to analytic singularities of μ on $]0, +\infty[$. Let $P_\cdot = -h^2\Delta + V_\cdot(x)$, where $V_\cdot \in C^\infty(\mathbf{R}^n; \mathbf{R})$, $\cdot = 0, 1$, and assume that the general assumptions in Chapter 8 are verified, with $\Phi_\cdot(\frac{1}{h^2}) = h^{-n}$. Then we can define μ as above.

Theorem 9.1 *Let $0 < E_0 \in \mathrm{sing\,supp}_a(\mu)$. Then for every complex neighborhood W of E_0, there exist $h_0 = h_0(W) > 0$, and $C = C(W) > 0$, such that for $h < h \leq h_0$, we have $\sum_0^1 \sharp(\mathrm{Res}\,(P_\cdot) \cap W) \geq \frac{1}{C(W)} h^{-n}$.*

The following corollary is stronger than the theorem:

Corollary 9.2 *Let $P_1 = -h^2\Delta + V_1(x)$, $V_1 \in C^\infty(\mathbf{R}^n; \mathbf{R})$ satisfy all the assumptions of the local trace formula (Th. 8.2) that make sense for one single operator. Let $0 < E_0 \in \mathrm{sing\,supp}_a(\nu_{+,1})$. Then for every complex neighborhood W of E_0, there exist $h_0 = h_0(W) > 0$, and $C = C(W) > 0$, such that for $h < h \leq h_0$, we have $\sharp(\mathrm{Res}\,(P_1) \cap W) \geq \frac{1}{C(W)} h^{-n}$.*

Proof of the Corollary. It suffices to construct $P_0 = -h^2\Delta + V_0(x)$, with $V_0 \in C_b^\infty(\mathbf{R}^n; \mathbf{R})$ such that (P_1, P_0) satisfies the assumptions of the local trace formula and such that:

$$\sup \mathrm{supp}\, \nu_{+,0} < E_0, \tag{9.6}$$

$$\exists \text{ a complex neighborhood } W_0 \text{ of } E_0 \text{ such that} \quad (9.7)$$
$$\operatorname{Res} P_0 \cap W_0 = \emptyset, \text{ when } h > 0 \text{ is small enough.}$$

We shall produce V_0 from V_1 by cut-off and regularization. Put $K(x) = C_n e^{-x^2/2}$ with $C_n > 0$ chosen so that $\int_{\mathbf{R}^n} K(x)dx = 1$. Put $K_\lambda(x) = \lambda^{-n}K(\lambda^{-1}x)$, for $\lambda > 0$. We make an x-dependent choice of λ:

$$\lambda(R, x) = R\langle R^{-1}x\rangle^{-N_0},$$

where $N_0 > 0$ is sufficiently large, depending on the dimension and $R \geq 1$ is a large parameter. Put $K_R(x, y) = K_{\lambda(R,x)}(x - y)$. Let $\chi \in C_0^\infty(B(0, 2); [0, 1])$ be equal to 1 on $B(0, 1)$. For R large enough, put

$$V_0(x) = \int K_R(x, y)(1 - \chi(R^{-1}y))V_1(y)dy.$$

Let $\theta > 0$ be small but independent of R. Then in the domain $|\operatorname{Im} x| < \theta\langle\operatorname{Re} x\rangle$:

$$|V_0(x)| \leq \epsilon(R),\ \epsilon(R) \to 0,\ R \to \infty,$$
$$|V_1(x) - V_0(x)| \leq C(R)\langle x\rangle^{-n-1}.$$

Then we have (9.6) and using that the resonances near E_0 can be viewed as eigenvalues of $P_{0|_{e^{i\theta/2}\mathbf{R}^n}}$, we see that (9.7) also holds. #

Proof of Theorem 9.1. We have seen that $\mathrm{WF}_a(\mu) = \mathrm{WF}_a(\omega)$ and since μ, ω are real, it is clear that $(E_0, 1), (E_0, -1) \in WF_a(\omega)$. Considering the definition of $\mathrm{WF}_a(\omega)$ by means of the FBI-transform ([73]), we see that there exist sequences $(\alpha_j, \beta_j) \to (E_0, 1)$ in $\mathbf{R}^2$, $\lambda_j \to +\infty$, $\epsilon_j \searrow 0$, such that

$$\left|\int e^{i\lambda_j(\beta_j(\alpha_j - E) + \frac{i}{2}(\alpha_j - E)^2)}\chi(E)\omega(E)dE\right| \geq e^{-\epsilon_j\lambda_j}, \quad (9.8)$$

where $\chi \in C_0^\infty(]0, \infty[)$ is equal to 1 near E_0 and has its support in a small neighborhood of E_0. Let a, b, a/b be small and positive and let

$$\Omega =]E_0 - b, E_0 + b[+i] - a, a],\ W =]E_0 - \frac{b}{2}, E_0 + \frac{b}{2}[+i] - \frac{a}{2}, a].$$

Let I, J be the intersections of W, Ω with the real line and choose $\chi \in C_0^\infty(J)$ equal to 1 near the closure of I. Let

$$f_j(E) = e^{i\lambda_j(\beta_j(\alpha_j - E) + \frac{i}{2}(\alpha_j - E)^2)}.$$

Then

$$|f_{j|_{\Omega\backslash W}}| \le e^{-\frac{1}{C_0}\lambda_j},$$

and (9.8) reads

$$|\int (\chi f_j)(E)\omega(E)dE| \ge e^{-\epsilon_j\lambda_j}. \tag{9.9}$$

The trace formula gives

$$''\mathrm{tr}\,[(\chi f_j)(P.)]_0^{1}{}'' = [\sum_{\mathrm{Res}\,(P.)\cap W_-} f_j(\lambda)]_0^1 + \mathcal{O}(h^{-n})e^{-\frac{1}{C_0}\lambda_j}. \tag{9.10}$$

On the other hand we have a trace formula of D. Robert [67], which in our situation implies:

$$\mathrm{tr}\,[(\chi f_j)(P.)]_0^1 = \frac{1}{(2\pi h)^n}\int (\chi f_j)(E)\omega(E)dE + \mathcal{O}_j(h^{1-n}). \tag{9.11}$$

(See the appendix of this chapter for an outline of a proof.)

Combining (9.10), (9.11), we get:

$$[\sum_{\mathrm{Res}\,(P.)\cap W_-} f_j(\lambda)]_0^1 = \frac{1}{(2\pi h)^n}\int (\chi f_j)(E)\omega(E)dE + \mathcal{O}(h^{-n})e^{-\frac{1}{C_0}\lambda_j} + \mathcal{O}_j(h^{1-n}),$$

so by (9.9) :

$$|[\sum_{\mathrm{Res}\,(P.)\cap W_-} f_j(\lambda)]_0^1| \ge \frac{e^{-\epsilon_j\lambda_j} - \mathcal{O}(1)e^{-\frac{1}{C_0}\lambda_j}}{(2\pi h)^n} + \mathcal{O}_j(h^{1-n}).$$

Choose first j large enough to see that the previous expression is

$$\ge \frac{\frac{1}{2}e^{-\epsilon_j\lambda_j}}{(2\pi h)^n} + \mathcal{O}_j(h^{1-n}).$$

Then fix j and choose h small enough to see that it can be bounded from below by $(2\pi h)^{-n}e^{-\epsilon_j\lambda_j}/3$. The theorem follows. #

We next consider resonances generated by analytic singularities on the negative axis.

Theorem 9.3 *We make the same general assumptions as in Theorem 9.1 and assume that the angle of scaling* θ_0 *is* $> \pi/2$. *Let* $0 > E_1 \in \operatorname{sing\,supp}_a(\mu)$. *Let* $\gamma : [0,1] \to \mathbf{C}$ *be a* C^1 *curve with* $\gamma(0) \in]\sup\operatorname{supp}(\mu), +\infty[$, $\gamma(1) = E_1$, $\operatorname{Im}\gamma(t) < 0$ *for* $0 < t < 1$. *Also assume that* γ *is injective and* $\gamma'(t) \neq 0$, $\forall t$. *Then for every neighborhood* W *of* $\gamma([0,1])$, *there exist constants* $C, h_0 > 0$, *such that*

$$\sum_0^1 \#(\operatorname{Res}(P_.) \cap W) \geq \frac{1}{C} h^{-n},\ \ 0 < h \leq h_0(W).$$

Corollary 9.4 *Let* $P_1 = -h^2\Delta + V_1(x)$ *satisfy all the assumptions of the local trace formula that make sense for one of the operators. Let the angle of scaling satisfy* $\theta_0 > \pi/2$, *and assume that* $V_1(x) = \mathcal{O}(\langle x\rangle^{-\widetilde{n}})$, *for some* $\widetilde{n} > n$ *in the region appearing in the corresponding complex scaling assumption. Let* $0 > E_1 \in \operatorname{sing\,supp}_a(\nu_{-,1})$, *and let* $\gamma : [0,1] \to \mathbf{C}$ *be a* C^1 *curve with* $\gamma(0) \in]\sup\operatorname{supp}(\nu_{+,1}), +\infty[$, $\gamma(1) = E_1$, $\operatorname{Im}\gamma(t) < 0$ *for* $0 < t < 1$, γ *injective and* $\gamma'(t) \neq 0$ *everywhere. Then for every neighborhood* W *of* $\gamma([0,1])$, *there exist* $C, h_0 > 0$ *such that*

$$\#(\operatorname{Res}(P_1) \cap W) \geq \frac{1}{C} h^{-n},\ \ 0 < h \leq h_0(W).$$

To get the corollary, it suffices to apply Theorem 9.3 with $P_0 = -h^2\Delta$. If we combine the two corollaries, we get the following general statement:

Consider $P_1 = -h^2\Delta + V_1(x)$ with angle of scaling $\theta_0 > \frac{\pi}{2}$ and with $V_1 = \mathcal{O}(\langle x\rangle^{-\widetilde{n}})$, $\widetilde{n} > n$ in the corresponding scaling region. If $V_1 \not\equiv 0$, then $\#(\mathrm{P}_1 \cap W) \geq \frac{1}{C(W)} h^{-n}$, $0 < h \leq h_0(W)$, where W is either any fixed complex neighborhood of some point on $]0, +\infty[$ or any neighborhood of a curve as in Corollary 9.4.

We refer to [77] for the proof of Theorem 9.3, and we here only explain some of the ideas. Let γ, W be as in the theorem and assume that W is open in $\{z \in \mathbf{C} \setminus \{0\};\ -2\theta_0 < \arg z \leq \epsilon_0\}$ for some small $\epsilon_0 > 0$ (as in the local trace formula). We may assume that the intersection of W with the real line is the union of two intervals $I_\pm \subset \mathbf{R}_\pm$, where $\mathbf{R}_\pm$ denote the open half-axes. Let $\Omega \supset\supset W$ have the same properties as W and let $J_\pm$ be the corresponding intervals. Let $\widehat{\chi} \in C_0^\infty(\Omega)$ be equal to 1 on $\overline{W}$, almost holomorphic at $J_\pm$. For a suitable sequence $f = f_j$ (that will not be constructed in detail here), with $|f| \leq 1$ on $\Omega \setminus W$, we write the trace formula:

$$[\sum_{\lambda \in \operatorname{Res} \cap W_-} f_j(\lambda)]_0^1 = {}''\operatorname{tr}[\chi_+ f_j(P_.)]_0^{1}{}'' + {}''\operatorname{tr}[\chi_- f_j(P_.)]_0^{1}{}'' + \mathcal{O}(h^{-n}).$$

Using the semiclassical trace formula, we get:

$$''\mathrm{tr}\,[\chi_\pm f_j(P_\cdot)]_0^1{}'' = \frac{1}{(2\pi h)^n}\int (\chi_\pm f_j)(E)\omega(E)dE + \mathcal{O}_j(h^{1-n}).$$

$\omega(E)$ extends analytically from J_+ to a function ω_+ on $\Omega\cap\{\mathrm{Im}\,z<0\}$ and by Stokes' formula, we get

$$\int \chi_+ f_j(E)dE = -\int (\chi_- f_j)(E)\omega_+(E-i0)dE + \mathcal{O}(1).$$

It follows that

$$[\sum_{\lambda\in\mathrm{Res}\,P_\cdot\cap W_-} f_j(\lambda)]_0^1 = \tag{9.12}$$
$$\frac{1}{(2\pi h)^n}\int (\chi_- f_j)(E)(\omega(E)-\omega_+(E-i0))dE + \mathcal{O}(h^{-n}) + \mathcal{O}_j(h^{1-n}).$$

Here $\omega(E)-\omega_+(E-i0)$ can be obtained from μ by convolution and we can show that $(E_1,\pm 1)$ both belong to the analytic wavefront set of $\omega(E)-\omega_+(E-i0)$.

Approximating Gaussian functions by suitable functions f_j as above, we can arrange so that the first term in the right hand side of (9.12) is $\gg \mathcal{O}(h^{-n})$ which permits us to conclude. #

We shall next consider clouds of resonances for systems, following the the article of L. Nedelec [59]. Schrödinger operators with matrix-valued potentials and similar systems appear frequently in quantum mechanics, for instance when one reduces the dimension by means of the Born-Oppenheimer reduction and similar methods that lead to a so called effective Hamiltonian. Although these effective Hamiltonians are not in general exactly equal to Schrödinger operators with matrix-valued potential, the latter form important models for more general systems that could also be studied with more or less the same methods.

Let

$$P = -h^2\Delta + V(x), \tag{9.13}$$

with $V(x)\in C_b^\infty(\mathbf{R}^n;\mathrm{Hermitian}\,r\times r\,\mathrm{matrices})$, satifying

(H1) $\exists\theta_0\in[0,\frac{\pi}{2}[$, $\epsilon>0$, $R>0$, such that V extends to a holomorphic function on $\{r\omega;\,\omega\in\mathbf{C}^n,\,\mathrm{dist}\,(\omega,S^{n-1})<\epsilon,\,r\in e^{i[0,\theta_0]}]R,+\infty[\}$.

(H2) There exists a Hermitian $r\times r$-matrix V_0 and a number $\widetilde{n}>n$ such that $\|V(x)-V_0\|\le C\langle x\rangle^{-\widetilde{n}}$ in the set in (H1).

Put $P_0=-h^2\Delta+V_0$, and let $\lambda_{1,0}\le,..\le\lambda_{r,0}$ be the eigenvalues of V_0, $x\in\mathbf{R}^n$. As in the scalar case, we can then define the resonances of P as the eigenvalues of the scaled operator $P_\theta=P_{|\Gamma_\theta}$, $0<\theta\le\theta_0$ in the set $\mathbf{R}\setminus\cup_{j=1}^r(\lambda_{j,0}+e^{-2i\theta}]0,+\infty[)$, and the resonances are independent of θ in the sence that for $0<\theta_1<\theta_2\le\theta_0$, $\theta_1\le\theta\le\theta_2$, and for every $1\le j\le r-1$, the eigenvalues of P_θ in the triangle bounded by the interval $]\lambda_{j,0},\lambda_{j+1,0}[$ and the half-rays $\lambda_{j,0}+e^{-2i\theta_1}]0,+\infty[$, $\lambda_{j+1,0}+e^{-2i\theta_2}]0,+\infty[$ are independent of θ. A similar statement holds in the sector $\lambda_{r,0}+e^{-2i[0,\theta_0[}]0,+\infty[$.

Assume $\lambda_{j,0}<\lambda_{j+1,0}$ for some $1\le j\le r$ and let $W\subset\subset\Omega$ be open and relatively compact in

$$(\lambda_{j,0}+e^{i]-2\theta_0,\epsilon_0]}]0,\infty[)\cap(]\lambda_{j,0},\lambda_{j+1,0}[+e^{-2i\theta_0}\mathbf{R}).$$

Here $\epsilon_0>0$ is small and we use the convention that $\lambda_{r+1,0}=+\infty$. Assume that $I:=W\cap\mathbf{R}$, $J:=\Omega\cap\mathbf{R}$ are intervals.

The local trace formula still holds:

Theorem 9.5 *Let $f=f(z;h)$ be holomorphic for $z\in\Omega$ with $|f(z;h)|\le 1$, $z\in\Omega\setminus W$. Let $1_I\prec\chi\in C_0^\infty(J)$ be independent of h. Let $P_1:=P$. Then $[\chi f(P.)]_0^1$ is of trace class and*

$$\mathrm{tr}\,[\chi f(P.)]_0^\infty=\sum_{z\in\mathrm{Res}\,(P)\cap W_-}f(z;h)+\mathcal{O}(h^{-n}),$$

where $W_-=W\cap e^{-2i[0,\theta_0[}]0,+\infty[$

The proof (see [59]) is essentially the same as in the scalar case.

On the other hand, we have Robert's trace formula:

Theorem 9.6 *Let $g\in C_0^\infty(\mathbf{R})$ be independent of h. Then $[g(P.)]_0^1$ is of trace class and*

$$\mathrm{tr}\,[g(P.)]_0^1=\frac{1}{(2\pi h)^n}\int\int\mathrm{tr}\,(g(p(x,\xi))-g(p_0(x,\xi)))dxd\xi+\mathcal{O}(h^{1-n}),$$

where $(x,\xi)=\xi^2+V(x)$, $p_0(x,\xi)=\xi^2+V_0(x)$.

Here the integral can also be written

$$\sum_j \int\int (g(\xi^2+\lambda_j(x)) - g(\xi^2+\lambda_{j,0}(x)))dxd\xi =: \langle \omega, g\rangle ,$$

where $\lambda_1(x) \le ... \le \lambda_r(x)$ denote the eigenvalues of $V(x)$. As in the scalar case,

$$\omega = C_n(E)_+^{\frac{n}{2}-1} * \mu, \text{ where } \langle \mu, \phi\rangle = \sum_j \int (\phi(\lambda_j(x)) - \phi(\lambda_{j,0}))dx.$$

For $E \in]\lambda_{j,0}, \lambda_{j+1,0}[$ we put

$$\nu_j(E) = -\sum_{k=1}^{j} \int_{\lambda_k(x)\ge E} dx + \sum_{k=j+1}^{r} \int_{\lambda_k(x)\le E} dx$$

Then on this interval, we have

$$\mu = \frac{d}{dE}\nu_j, \text{ sing supp}_a(\omega) = \text{sing supp}_a(\mu) = \text{sing supp}_a(\nu_j).$$

The arguments in the scalar case extend (see [59]) and give

Theorem 9.7 *Let* $]\lambda_{j,0}, \lambda_{j+1,0}[\ni E_0 \in \text{sing supp}_a(\nu_j)$*. Then for any neighborhood* W *of* E_0*, there exist* $h_0, C > 0$*, such that*

$$\#(\text{Res}\,(P) \cap W) \ge \frac{1}{C} h^{-n}, \ 0 < h \le h_0.$$

If $\lambda_k(x)$ is a simple eigenvalue near some point x_0, then $\lambda_k(x)$ is a smooth function there, and if $E_0 = \lambda_k(x_0)$ is a critical value, we expect in general that E_0 will belong to the analytic singular support of μ. This is essentially as in the scalar case. Multiple eigenvalues can also give rise to analytic singularities of μ and we now describe such a situation:

Let $E_0 \in]\lambda_{j,0}, \lambda_{j+1,0}[$, $1 \le j \le r$, with the convention $\lambda_{r+1,0} = +\infty$, and assume:

0) $n = 2$, V is real and symmetric.

1) For $x = 0$, there are exactly two eigenvalues $\lambda_k(0), \lambda_{k+1}(0)$ that are equal to E_0.

2) If $E_0 = \lambda_\nu(x)$ for some ν and some $x \neq 0$, then $\lambda_\nu(x)$ is a simple eigenvalue, and $d\lambda_\nu(x) \neq 0$.

3) Let $a(x)I+\begin{pmatrix} c(x) & b(x)\\ b(x) & -c(x)\end{pmatrix}$ be a smooth representative of the restriction of $V(x)$ to the 2-dimensional eigenspace corresponding to the two eigenvalues $\lambda_k(x),\lambda_{k+1}(x)$, for x in a neighborhood of 0. Then $dc(0),db(0)$ are linearly independent and $da(0)=\beta db(0)+\gamma dc(0)$ with $\beta^2+\gamma^2<1$. Under these assumtions it is proved in [59] that $E_0\in \mathrm{Sing\,supp}_a\mu$.

Appendix.

We first recall the operator version of the Cauchy-Riemann-Green-Stokes formula, which was used (in a slightly different operator version) by Dynkin [22] and has been used very much since the work [39]. Let $P:\mathcal{H}\to\mathcal{H}$ be some unbounded selfadjoint operator, and let $f\in C_0^\infty(\mathbf{R})$. Let $\widetilde{f}\in C_0^\infty(\mathbf{C})$ be an almost holomorphic extension of f with the property that $\overline{\partial}\widetilde{f}$ vanishes to infinite order on the real axis). Then since $1/(\pi z)$ is a fundamental solution of the Cauchy-Riemann operator $\overline{\partial}$, we have

$$f(E)=-\frac{1}{\pi}\int\overline{\partial}\widetilde{f}(z)\frac{1}{z-E}L(dz),\ \ E\in\mathbf{R}. \tag{A.1)}$$

The operator version follows by replacing "E" by "P":

$$f(P)=-\frac{1}{\pi}\int\overline{\partial}\widetilde{f}(z)(z-P)^{-1}L(dz),\ \ E\in\mathbf{R}. \tag{A.2)}$$

The proof is a straight forward application of the spectral theorem. ([39], [21], [20]).

Let $P_\cdot$ be as in the main text of this chapter. Let $\chi\in C_0^\infty(\mathbf{R})$ be conveniently chosen, so that the infimum of the spectrum of $\widetilde{P}_\cdot:=-h^2\Delta+V_\cdot(x)+\chi(hD)$ is $\geq 1+\sup\mathrm{supp}\,f$. The resolvent identity gives

$$(z-P_\cdot)^{-1}=(z-\widetilde{P}_\cdot)^{-1}+(z-\widetilde{P}_\cdot)^{-1}\chi(hD)(z-P_\cdot)^{-1}. \tag{A.3}$$

Using this in (A.2) we get, since $f(\widetilde{P}_\cdot)=0$:

$$f(P_\cdot)=-\frac{1}{\pi}\int\overline{\partial}\widetilde{f}(z)(z-\widetilde{P}_\cdot)^{-1}\chi(hD)(z-P_\cdot)^{-1}L(dz). \tag{A.4}$$

As for instance in the proof of the invariance of the condition (7.12), we see that $[(z-\widetilde{P}_\cdot)^{-1}\chi(hD)(z-P_\cdot)^{-1}]_0^1$ is of trace class with corresponding norm $\leq\mathcal{O}(1)h^{-n}|\mathrm{Im}\,z|^{-N_0}$, for $z\in\mathrm{neigh\,supp}\,(\widetilde{f})$. A convenient way to see this, is to put $P_t=(1-t)P_0+tP_1$, and to write

$$[(z-\widetilde{P}_\cdot)^{-1}\chi(hD)(z-P_\cdot)^{-1}]_0^1=\int_0^1\frac{\partial}{\partial t}((z-\widetilde{P}_t)^{-1}\chi(hD)(z-P_t)^{-1})dt, \tag{A.5}$$

then develop the right hand side and use the same arguments as in Chapter 7.

It follows that for any fixed $\delta > 0$:

$$\mathrm{tr}\,[f(P.)]_0^1 = -\frac{1}{\pi}\mathrm{tr}\int_{|\mathrm{Im}\,z|>h^\delta} \overline{\partial}\widetilde{f}(z)[(z-\widetilde{P}.)^{-1}\chi(hD)(z-P.)^{-1}]_0^1 L(dz)+\mathcal{O}(h^\infty). \tag{A.6}$$

Let $0 < \delta < \frac{1}{2}$. Then $(z - P_t)^{-1}$ is an h-pseudodifferential operator with symbol a of class $S_\delta(h^{-\delta}\langle\xi\rangle^{-2})$ for $|\mathrm{Im}\,z| > h^\delta$, in the sense that

$$\partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(h^{-\delta(|\alpha|+|\beta|)}h^{-\delta}\langle\xi\rangle^{-2}).$$

(The detailed proof of this uses a semi-classical version of a lemma of R. Beals, see [39] and also [21].) In this class, the symbol has an asymptotic expansion:

$$\sim \frac{1}{(z-p_t(x,\xi))} + \frac{ha_1(x,\xi,z)}{(z-p_t)^3} + \frac{h^2a_2(x,\xi,z)}{(z-p_t)^5} + ...,.$$

[[Nästa version, skriv lite mer om egenskaperna hos a_j.]] (See [21].) Using this in (A.5), we see that $[(z-\widetilde{P}.)^{-1}\chi(hD)(z-P.)^{-1}]_0^1$ has a symbol of class $S_\delta(h^{-2\delta}\langle\xi\rangle^{-N}\langle x\rangle^{-\widetilde{n}})$ for every $n > 0$. The symbol is equal to

$$[(z-\widetilde{p}.)^{-1}\chi(\xi)(z-p.)^{-1}]_0^1 + \int \mathcal{O}(h)\langle x\rangle^{-\widetilde{n}}\langle\xi\rangle^{-N}|z-p_t|^{-N_0}dt. \tag{A.7}$$

We plug this into (A.6) and use that an h-pseudodifferential operator with symbol a in $S_\delta(m)$ is of trace class if m is integrable, and has the trace $\frac{1}{(2\pi h)^n}\int\int a(x,\xi)dxd\xi$. It follows that

$$\mathrm{tr}\,[f(P.)]_0^1 = -\frac{1}{\pi}\int\int\int \overline{\partial}\widetilde{f}(z)[(z-\widetilde{p}.)^{-1}\chi(\xi)(z-p.)^{-1}]_0^1\frac{dxd\xi}{(2\pi h)^n}L(dz)+\mathcal{O}(h^{1-n}). \tag{A.8}$$

Here

$$\begin{aligned}&-\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)[(z-\widetilde{p}.(x,\xi))^{-1}\chi(\xi)(z-p.)]_0^1 L(dx) =\\ &-\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)[(z-p.)^{-1}-(z-\widetilde{p}.)^{-1}]_0^1 L(dx) =\\ &[f(p.(x,\xi))-f(\widetilde{p}.(x,\xi))]_0^1 = [f(p.(x,\xi))]_0^1,\end{aligned}$$

since $f(\widetilde{p}.(x,\xi)) = 0$ by construction.

It follows that

$$\mathrm{tr}\,[f(P_\cdot)]_0^1 = \frac{1}{(2\pi h)^n}\int\int [f(p_\cdot(x,\xi))]_0^1 dx d\xi + \mathcal{O}(h^{1-n}),$$

as claimed.

10 Resonances generated by closed trajectories.

In this chapter we show how closed classical trajectories can give rise to singularities in the local trace-formula, leading to the existence of many resonances. Here enters also some version of the so called Gutzwiller trace formula. This strategy was used by Bardos-Lebeu-Rauch [7], in order to show that there are infinitely many resonances associated to certain types of strictly convex obstacles, and later by Ikawa [45] in the case of an obstacle composed by several strictly convex disjoint parts. Quantitative results with lower bounds on some counting function for the resonances were obtained by Sjöstrand and Zworski [84]. The lower bounds in [84] were originally obtained by using the exact Bardos-Guillot-Ralston trace-formula, and hence limited to the case of odd space-dimensions, however the local trace-formula can be used here as well ([75]), and as a consequence we avoid the restriction on the dimension. Here we prefer to discuss an analogous recent result by J.F. Bony [11], which treats explicitly the semi-classical situation. It is also a bit sharper than the corresponding result in [84].

Let

$$P = \sum_{|\alpha|\le 2} a_\alpha(x;h)(hD_x)^\alpha :\ L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n) \tag{10.1}$$

satisfy the general assumptions of Chapter 7. Assume for simplicity that

$$a_\alpha(x;h)\sim \sum_{j\ge 0} a_\alpha^j(x)h^j \tag{10.2}$$

in C_b^∞ and that this also holds in the complex domain (as in (7.41)). Let

$$p = p_0(x,\xi) = \sum_{|\alpha|\le 2} a_\alpha^0(x)\xi^\alpha \tag{10.3}$$

be the semi-classical principal symbol of P. Let

$$H_p = \sum_{j=1}^{n} (\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j}) \tag{10.4}$$

be the corresponding Hamilton field.

Let $E_0 > 0$, $T_0 > 0$ and assume that $\gamma : \mathbf{R} \ni t \mapsto \gamma(t) \in p^{-1}(E_0)$ is a T_0-periodic H_p-trajectory along which $dp \neq 0$. Let us recall some properties of the associated *Poincaré map*: At a point $\rho_0 = \gamma(t_0)$, the space $T_{\rho_0}(p^{-1}(E_0))/\mathbf{R}H_p(\rho_0)$ is a symplectic vectorspace: If $\sigma = \sum d\xi_j \wedge dx_j$ is the standard symplectic form that we also view as a bilinear form: $\langle \sigma, t \wedge s \rangle = t_\xi \cdot s_x - t_x \cdot s_\xi$, $t = (t_x, t_\xi)$, $s = (s_x, s_\xi)$, then the symplectic orthogonal space of $T_{\rho_0}(p^{-1}(E_0))$ is $\mathbf{R}H_p \subset T_{\rho_0}(p^{-1}(E_0))$ and we get a natural non-degenerate alternate bilinear form on $T_{\rho_0}(p^{-1}(E_0))/\mathbf{R}H_p(\rho_0)$. A more or less equivalent version of this, is that if $\Sigma \subset p^{-1}(E_0)$ is a hypersurface (i.e. of dimension $(2n-1)-1 = 2(n-1)$) passing through ρ_0 transversally to $H_p(\rho_0)$, then near ρ_0, $(\Sigma, \sigma_{|_\Sigma})$ is a symplectic manifold.

The (non-linear) Poincaré map is then obtained in the following way: Start at $\rho \in \Sigma \cap \mathrm{neigh}\,(\rho_0)$ and follow the H_p-trajectory until you hit Σ again for a unique time $t(\rho) = T_0 + \mathcal{O}(|\rho - \rho_0|)$, at the point $\kappa(\rho) = \exp(t(\rho)H_p)(\rho)$. κ is then the non-linear Poincaré map. (If $\widetilde{\rho}_0 = \gamma(s_0)$ is a different base point, and $\widetilde{\Sigma} \subset p^{-1}(E_0)$ a hypersurface transversal to $H_p(\widetilde{\rho}_0)$, let $\widetilde{\kappa}$ be the corresponding Poincaré map. Then

$$\kappa = \alpha^{-1} \circ \widetilde{\kappa} \circ \alpha,$$

where $\alpha : \Sigma \cap \mathrm{neigh}\,(\rho_0) \to \widetilde{\Sigma} \cap \mathrm{neigh}\,(\widetilde{\rho}_0)$ is given by $\alpha(\rho) = \exp(r(\rho)H_p(\rho)$, with $t(\rho) = s_0 - t_0 + \mathcal{O}(|\rho - \rho_0|)$, . Then it is easy to check that the non-linear Poincaré map $\kappa : \Sigma \cap \mathrm{neigh}\,(\rho_0) \to \Sigma \cap \mathrm{neigh}\,(\rho_0)$ is a canonical transformation (or symplectomorphism, i.e. a map which conserves the symplectic form) and the corresponding linearized Poincaré map $d\kappa(\rho_0) : T_{\rho_0}\Sigma \to T_{\rho_0}\Sigma$ will of course be a symplectomorphism also. Notice that $T_{\rho_0}\Sigma \simeq T_{\rho_0}(p^{-1}(E_0))/(\mathbf{R}H_p(\rho_0))$. Since $d\kappa(\rho_0)$ is a real symplectomorphism, we know that its eigenvalues can be grouped into the following families:

1) The value 1 with even multiplicity,

2) The value –1 with even multiplicity.

3) The values λ and λ^{-1} for some $\lambda \in]-\infty, -1[\cup]1, +\infty[$

4) The values $\lambda, \overline{\lambda}$ for some λ with $|\lambda| = 1$, $\mathrm{Im}\,\lambda > 0$.

5) The values $\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$, for some $\lambda \in \mathbf{C}$ with $\Im\lambda > 0$, $|\lambda| > 1$.

For each of the groups in 3)–5), the eigenvalues have the same multiplictity.

Assume that the (linearized) Poincaré map is non-degenerate in the sense that 1 is not an eigenvalue of $d\kappa(\rho_0)$. Then by the implicit function theorem, applied to the map $\rho \mapsto \kappa(\rho) - \rho$, we see that ρ_0 is an isolated fixed point of κ and this situation is stable under small perturbations of $p - E_0$. We can therefore vary $E \in \mathrm{neigh}\,(E_0, \mathbf{R})$ and see that for each such energy, there is a unique closed trajectory γ_E in $p^{-1}(E)$ of period $T(E) = T_0 + \mathcal{O}(E - E_0)$. Moreover this trajectory depends smoothly on E in the obvious sense.

The set $\mathcal{S} = \cup_{E \in \mathrm{neigh}\,(E_0, \mathbf{R})} \mathcal{R}(\gamma_E)$ (where "$\mathcal{R}$") stands for "range of", is a smooth symplectic 2-dimensional manifold. Write $\mathcal{P} = d\kappa_E(\rho_E)$, where $\rho_E \in \mathcal{R}(\gamma_E)$ and κ_E denote a fixed point and Poincaré map at energy E.

Assume that $\gamma_0 = \gamma_{E_0}$ is (up to trivial reparametrizations) the only T_0 periodic H_p-trajectory in $p^{-1}(E_0)$. (This assumption can be weakened to allow for finitely many trajectories, obtained from γ_0 by the action of some finite symmetry group.) Notice that this assumption implies that E_0 is not a critical value of p.

Under the above assumptions we have the following theorem, due to J.F. Bony [11].

Theorem 10.1 *Let $\epsilon, \mu > 0$. Then $\exists C > 0$ such that for every interval $[a(h), b(h)] \subset [E_0 - \frac{1}{C}, E_0 + \frac{1}{C}]$, $0 < h \leq \frac{1}{C}$, we have*

$$\sum_{\lambda \in \mathrm{Res}\, P \cap W} e^{T(\mathrm{Re}\,\lambda)(\mathrm{Im}\,\lambda)(1-\mu)/h} \geq \tag{10.5}$$

$$\frac{1}{2\pi h} \int_{a(h)}^{b(h)} T^*(E) |\det(1 - \mathcal{P}_{\gamma_E})|^{-1/2} dE - \mathcal{O}(h^{1/C}) \frac{b(h) - a(h)}{h},$$

where

$$W = \{\lambda \in \mathbf{C}; a(h) - Ch \log \frac{1}{h} \leq \mathrm{Re}\,\lambda \leq b(h) + Ch \log \frac{1}{h},$$
$$-\frac{h \log \frac{1}{h}}{T(\mathrm{Re}\,\lambda)} (n - 1 + \epsilon \frac{\log(b(h) - a(h))}{\log h}) \leq \mathrm{Im}\,\lambda \leq 0\}$$

and $T^*(E) > 0$, is the primitive period, i.e. the smallest positive period of γ_E.

Corollary 10.2 *Under the same assumptions, we have the same lower bound on $\#(\mathrm{Res}\,(P) \cap W)$.*

Outline of the proof. Let P_j be two operators as above, with $[a_{\alpha,j}]_0^1 = \mathcal{O}(1)\langle x\rangle^{-\widetilde{n}}$, for some $\widetilde{n} > n$. We have then an extension of the local trace formula: Let

$$\Omega =]-b, b[+i]-a, a], \; W =]-\frac{b}{2}, \frac{b}{2}[+i]-\frac{a}{2}, a], \; a, b > 0.$$

Let I, J be the intersections of W, Ω with the real line and choose $\chi \in C_0^\infty(J)$ equal to 1 near the closure of I. Put $\Omega_\delta = E_0 + \delta\Omega$, $W_\delta = E_0 + \delta W$, $\chi_\delta(E) = \chi(\frac{E-E_0}{\delta})$.

Theorem 10.3 *Let $Ch < \delta < 1/C$, with $C \gg 1$ sufficiently large. Let $f(z; h)$ be holomorphic in Ω_δ with $|f(z; h)| \leq 1$ for $z \in \Omega_\delta \setminus W_\delta$. Then*

$$\operatorname{tr}[(\chi_\delta f)(P_\cdot; h)]_0^1 = [\sum_{\lambda \in \operatorname{Res} P} f(\lambda; h)]_0^1 + \mathcal{O}(1)\delta^{-1}h^{-n}. \tag{10.6}$$

The proof is an adaptation of the usual one. See [11]. Another basic ingredient is the Gutzwiller trace formula, which in this version was obtained by D. Robert [67] and to which J.F. Bony [11] has given a detailed proof, involving the approximation of $e^{-itP_\cdot/h}$ by Fourier integral operators.

We now describe that formula in the special case that we will need it. Let $A > 0$ be small and let $g(t) \in C_0^\infty(\operatorname{neigh}(T([E_0 - A, E_0 + A]); \mathbf{R}))$ be equal to 1 near $T([E_0 - A, E_0 + A])$. Put

$$f_g(F) = f_{g,C,E}(F) = \int e^{-it(F-E)/h} g(t) e^{-(t-T(E))^2 C(\log h^{-1})/2} dt.$$

Let

$$f(F) = f_1(F) = \frac{\sqrt{2\pi}}{\sqrt{C \log h^{-1}}} e^{-iT(E)(F-E)/h - \frac{(F-E)^2}{2Ch^2 \log(h^{-1})}}.$$

Let $1_{[E_0-2A, E_0+2A]} \prec \chi \in C_0^\infty(]E_0 - 3A, E_0 + 3A[)$, and put $P_1 = P$. It is possible to construct P_0 as in the local trace-formula, so that P_0 has no resonances in a fixed h-independent neighborhood of $[E_0 - A, E_0 + A]$, and so that the principal symbol p_0 has no closed classical trajectories with energy in $[E_0 - 3A, E_0 + 3A]$ with period in some fixed neighborhood of $T([E_0 - 3A, E_0 + 3A])$

Theorem 10.4 *For $E \in [E_0 - A, E_0 + A]$ and h small enough, we have*

$$\operatorname{tr}[(\chi^2 f_g)(P_j)]_0^1 = \\ e^{iS(\gamma_E)/h + i(\sigma(\gamma_E)\pi/4 - S_1(\gamma_E))} T^*(E) |\det(1 - \mathcal{P}_{\gamma_E})|^{-\frac{1}{2}} + \mathcal{O}(h \log h^{-1}).$$

Here $S(\gamma_E) = \int_{\gamma_E} \xi \cdot dx$ *is the action of* γ_E *and* $\sigma(\gamma_E)$ *and* $S_1(\gamma_E)$ *are real-valued.*

We can now end the outline of the proof of Theorem 10.1. Combining the last two theorems we get

$$\sum_{\lambda \in \mathrm{Res}\,(P) \cap W_{E,h}} |f_{g,E}(\lambda)| \geq T^*(E) |\det(1 - \mathcal{P}_{\gamma_E})|^{-1/2} - \mathcal{O}(h \log \frac{1}{h}).$$

[[Här blir mitt handskrivna manus alltför otydligt. Skall återkomma till detta senare! Beviset avslutas i alla fall med att man integrerar olikheten m.a.p. E.]]

11 From quasi-modes to resonances.

We shall describe a result of P. Stefanov which culminates a series of works, started by Stefanov–Vodev [88] and continued by Tang–Zworski [89]. Actually many of the basic ideas can be traced back much further to works of Agmon, Agranovich and others in connection with the completeness of the set of generalized eigenfunctions for non-selfadjoint operators.

Let $P = P(h) : \mathcal{H} \to \mathcal{H}$ be a a semiclassical black box operator as in chapter 7, for which we can define the resonances in $e^{i]-2\theta_0,0]}]0, +\infty[$. As already mentioned in that chapter, we can also define the resonances as the poles of the meromorphic extension from the upper half-plane across $]0, +\infty[$ of $(z - P)^{-1} : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$.

We need some remarks about truncated spectral projections. Let

$$1_{B(0,R_0)} \prec \chi \in C_0^\infty(\mathbf{R}^n)$$

be such that "the exterior" operator Q has analytic coefficients in a neighborhood of $\mathrm{supp}\,(1 - \chi)$. We choose Γ_θ so that $\Gamma_\theta = \mathbf{R}^n$ near $\mathrm{supp}\,\chi$. Let $z_0 \in e^{i]-2\theta,0]}]0. + \infty[$ be a resonance, $0 < \theta \leq \theta_0$. Let

$$\pi_{\theta,z_0} = \frac{1}{2\pi i} \int_{\partial D(z_0,\epsilon)} (z - P_\theta)^{-1} dz, \ P_\theta = P_{|\Gamma_\theta}, \ 0 < \epsilon \ll 1, \tag{11.1}$$

be the corresponding spectral projection. We know that $\mathrm{rank}\,\pi_{\theta,z_0}$ is independent of the choice of θ with $-\frac{1}{2}\arg z_0 < \theta \leq \theta_0$.

Lemma 11.1 $\mathrm{rank}\,(\chi \pi_{\theta,z_0}) = \mathrm{rank}\,(\pi_{\theta,z_0})$.

Proof. We know that $\mathcal{R}(\pi_{\theta,z_0}) = \mathcal{N}((P_\theta - z_0)^{N_0})$ for some $1 \le N_0 \le m(z_0)$. Let $u \in \mathcal{R}(\pi_{\theta,z_0})$ with $\chi u = 0$. From the extension lemma 7.2, we conclude that $u = 0$. It follows that $\mathrm{rank}\,(\pi_{\theta,z_0}) \le \mathrm{rank}\,(\chi\pi_{\theta,z_0})$, and the lemma follows since the opposite inequality is obvious. #

Recall that $\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0,R_0))$. Let $\Gamma_\theta = \overline{\Gamma}_\theta$. If $u \in L^2(\Gamma_\theta)$, then $u^*(x) := \overline{u(\overline{x})} \in L^2(\Gamma_{-\theta})$, and we see that $L^2(\Gamma_{-\theta})$ and $L^2(\Gamma_\theta)$ become mutually dual spaces for the scalar product $(u|v) = \int_{\Gamma_\theta} u(x)v^*(x)dx$. This extends in the natural way to a duality between $\mathcal{H}_\theta$ and $\mathcal{H}_{-\theta}$, and $P_{-\theta} : \mathcal{H}_\theta \to \mathcal{H}_{-\theta}$ becomes the adjoint of P_θ. If z_0 is an eigenvalue of P_θ, then $\overline{z}_0$ is an eigenvalue of $P_{-\theta}$, and we get $\pi_{-\theta,\overline{z}_0} = \pi^*_{\theta,z_0}$ for the corresponding spectral projections. It follows that

$$\mathrm{rank}\,(\pi_{\theta,z_0}\chi) = \mathrm{rank}\,(\overline{\chi}\pi_{-\theta,\overline{z}_0}) = \mathrm{rank}\,(\pi_{-\theta,\overline{z}_0}) = \mathrm{rank}\,(\pi_{\theta,z_0}).$$

Hence $\mathcal{R}(\pi_{\theta,z_0}) = \mathcal{R}(\pi_{\theta,z_0}\chi)$, and we get:

$$\mathrm{rank}\,(\pi_{\theta,z_0}) = \mathrm{rank}\,(\chi\pi_{\theta,z_0}) = \mathrm{rank}\,(\pi_{\theta,z_0}\chi) = \mathrm{rank}\,(\chi\pi_{\theta,z_0}\chi). \tag{11.2}$$

Let $R(z) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$ denote the meromorphic extension of $(z-P)^{-1}$ from $e^{i]0,\epsilon_0[}]0,+\infty[$ to $e^{i]-2\theta_0,\epsilon_0[}]0,+\infty[$. From Chapter 7 it follows that if z_0 is a resonance, then

$$\mathrm{rank}\,\pi_{z_0} \le \mathrm{rank}\,\pi_{\theta,z_0},\ -\frac{1}{2}\mathrm{arg}\,z_0 < \theta \le \theta_0, \tag{11.3}$$

where

$$\pi_{z_0} = \frac{1}{2\pi i}\int_{\partial D(z_0,\epsilon)} R(z)dz.$$

On the other hand, since $\chi(z-P_\theta)^{-1}\chi = \chi R(z)\chi$, it is clear that

$$\chi\pi_{z_0}\chi = \chi\pi_{\theta,z_0}\chi, \tag{11.4}$$

so from (11.2), (11.3), we get

$$\mathrm{rank}\,(\pi_{z_0}) = \mathrm{rank}\,(\chi\pi_{z_0}\chi) = \mathrm{rank}\,(\pi_{\theta,z_0}) = \mathrm{rank}\,(\chi\pi_{\theta,z_0}\chi). \tag{11.5}$$

Recall that among our assumptions we have (7.12):

$$N(P^\sharp, [-\lambda,\lambda]) = \mathcal{O}(1)\Phi(\frac{\lambda}{h^2}),\ \lambda \ge 1.$$

We also made assumptions on Φ which imply that $\Phi(t) = \mathcal{O}(t^{n^\sharp/2})$ for some $n^\sharp \geq n$. The following result is a simple long-range extension of a result of P. Stefanov [87]. (The main novelty of Stefonov's result compared to that of Tang-Zworski [89] was that Stefanov is able to treat multiplicities in the case when quasi-modes are very close to each other.)

Theorem 11.2 *Let $H \subset]0, h_0]$ have 0 as an accumulation point. Let $0 < a_0 \leq a(h) \leq b(h) \leq b_0 < \infty$ be two functions on H. Assume that for every $h \in H$, $\exists\, m(h) \in \{1, 2, ...\}$ and $E_j(h) \in [a(h), b(h)]$, $u_j(h) \in \mathcal{D}$, $1 \leq j \leq m(h)$, such that* $\operatorname{supp} u_j \in K \subset\subset \mathbf{R}^n$, *where K is independent of h,*

$$\|(P(h) - E_j(h))u_j(h)\|_{\mathcal{H}} \leq R(h),\ 1 \leq j \leq m(h),$$

$$|(u_i(h)|u_j(h)) - \delta_{i,j}| \leq R(h),$$

where $R(h) = \mathcal{O}(h^\infty)$. Let $S(h)$ be a function with

$$\max(h^{-n^\sharp - 1} R(h), e^{-D/h}) \leq S(h) = \mathcal{O}(h^\infty),$$

for some fixed constant $D > 0$.

Then for every $k \in \mathbf{N}$, $\exists h(S, k) > 0$, such that for $H \ni h < h(S, k)$, $P(h)$ has at least $m(h)$ resonances in

$$[a(h) - 6h^k, b(h) + 6h^k] + i[0, 2S(h)h^{-1-n^\sharp}].$$

The difficulty is that the problem is non-selfadjoint. Under the same assumptions, it is much easier to prove that $P^\sharp(h)$ has at least $m(h)$ eigenvalues in $[a(h) - h^k, b(h) + h^k]$. In particular, we know that $m(h) = \mathcal{O}(h^{-n^\sharp})$. The following lemma is essentially from [89].

Lemma 11.3 *For z in some fixed compact subset of $e^{i]-2\theta_0, 0]}]0, \infty[$, there exists $A > 0$ such that we have*

$$\|\chi(z - P(h))^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \leq A e^{Ah^{-n^\sharp} \log g(h)^{-1}},$$

if $\operatorname{dist}(z, \operatorname{Res} P(h)) \geq g(h) > 0$. *(Here $(z - P(h))^{-1}$ denotes the meromorphic extension of the resolvent, when* $\operatorname{Im} z \leq 0$.*)*

Proof. Recall from Chapter 8 that

$$z - P_\theta = (1 + \widetilde{K}(z))(z - \widehat{P}_\theta),\ \theta = \theta_0,$$

where $\|\widetilde{K}\| = \mathcal{O}(1)$. Also recall that $\chi(z-P(h))^{-1}\chi = \chi(z-P_\theta)^{-1}\chi$. It is therefore enough to show that

$$\|(1+\widetilde{K}(z))^{-1}\| \leq Ae^{Ah^{-n^\sharp}\log\frac{1}{g}}, \text{ when } \operatorname{dist}(z, \operatorname{Res} P) \geq g(h). \tag{11.6}$$

Recall that

$$\det(1+\widetilde{K}(z)) = GD_w,$$

where $|G| \geq A^{-1}e^{-Ah^{-n^\sharp}}$ It is easy to see that $|D_w| \geq A^{-1}e^{-Ah^{-n^\sharp}\log\frac{1}{g}}$, and we get (11.6) with a new constant A. #

Notice that $\|\chi(z-P)^{-1}\chi\| \leq 1/|\operatorname{Im} z|$, for $\operatorname{Im} z > 0$. We state the following consequence of the maximum principle essentially from [89].

Lemma 11.4 *For $h \in H$, let $F(z;h)$ be holomorphic near $\Omega_5(h) = [E(h) - 5h^k, E(h)+5h^k] + i[-S(h)h^{-1-n^\sharp}, S(h)]$, with $E(h) \in \mathbf{R}$, $S(h)$ as in the theorem. Assume that*

$$|F(z;h)| \leq A\exp(Ah^{-n^\sharp}\log\frac{1}{hS(h)}) \text{ on } \Omega_5(h),$$

$$|F(z;h)| \leq \frac{1}{|\operatorname{Im} z|} \text{ on } \Omega_5(h) \cap \{z; \operatorname{Im} z > 0\}.$$

Then there exist $h_1 = h_1(S,A,k) > 0$, $B = B(S,A,k) > 0$ (independent of h) such that

$$|F(z;h)| \leq \frac{B}{S(h)},\ z \in [E(h)-h^k, E(h)+h^k],\ h \in H,\ h \leq h_1.$$

Proof. This is a routine argument for subharmonic functions. Consider the subharmonic function

$$f(z) = \log|F(z;h)| - \log(\frac{1}{S(h)})\frac{\operatorname{Im} z + S(h)h^{-1-n^\sharp}}{S(h)+S(h)h^{-1-n^\sharp}}$$
$$-((\log A) + Ah^{-n^\sharp}\log(\frac{1}{hS(h)}))\frac{-\operatorname{Im} z + S(h))}{(S(h)+S(h)h^{-1-n^\sharp}},$$

which is ≤ 0 on the horizontal segments of $\partial\Omega_5(h)$ and satisfies

$$|f(z)| \leq (\log A) + Ah^{-n^\sharp}\log(\frac{1}{hS(h)})$$

on the vertical segments.

After the change of coordinates, $z = (S + Sh^{-1-n^\sharp})w$, Ω_5 becomes a rectangle of vertical "width" 1 and of horizontal "length" $\frac{10h^k}{S(1+h^{-1-n^\sharp})}$. In such a long rectangle, the Poisson kernel decays exponentially (see for instance [76]), and it follows that in the rectangle of the same center and same width and of length $\frac{2h^k}{S(1+h^{-1-n^\sharp})}$, we have

$$\begin{aligned}|f(z)| &\leq ((\log A) + Ah^{-n^\sharp} \log(\frac{1}{hS}))e^{-h^k/(S(1+h^{-1-n^\sharp}))} \\ &\leq \mathcal{O}(h^\infty) \log(\frac{1}{hS}) \leq \log \frac{1}{S}.\end{aligned}$$

Restricting this further to real z, we get

$$\log |F(z; h)| \leq \mathcal{O}(1) \log \frac{1}{S},$$

and the lemma follows. #

Let z_0 be a resonance. Then we know that for $z \in \operatorname{neigh}(z_0)$:

$$(z - P_\theta)^{-1} = \sum_{j=1}^{N} \frac{\pi^{(j)}_{z_0,\theta}}{(z - z_0)^j} + \operatorname{hol}(z),$$

where

$$\pi^{(1)}_{z_0,\theta} = \pi_{z_0,\theta}$$

is the spectral projection and $\mathcal{R}(\pi^{(j)}_{z_0,\theta}) \subset \mathcal{R}(\pi_{z_0,\theta}) = \mathcal{R}(\pi_{z_0,\theta}\chi)$, $j \geq 1$, if we choose χ as in the beginning of this chapter. It follows that $\mathcal{R}(\chi\pi^{(j)}_{z_0,\theta}\chi) \subset \mathcal{R}(\chi\pi_{z_0,\theta}\chi)$, and recalling that $\chi(z - P)^{-1}\chi = \chi(z - P_\theta)^{-1}\chi$, we get

$$\chi(z - P)^{-1}\chi = \sum_{j=1}^{N} (z - z_0)^{-j} A_j + \operatorname{hol}(z),\ z \in \operatorname{neigh}(z_0), \tag{11.7}$$

with $A_j = \chi\pi^{(j)}_{z_0,\theta}\chi$, $\mathcal{R}(A_j) \subset \mathcal{R}(A_1)$, $\operatorname{rank}(A_1) = m(z_0)$.

End of the proof of the theorem. Let $z_1(h), ..., z_{M(h)}(h)$, be the distinct resonances of $P(h)$ in

$$\Omega_6(h) := [a(h) - 6h^k, b(h) + 6h^k] + i[-2S(h)h^{-1-n^\sharp}, S(h)],$$

and let

$$A^{(j)}(h) = A_{1,z_j}(h) = \frac{1}{2\pi i}\int_{\partial D(z_j,\epsilon)} \chi(z-P)^{-1}\chi dz.$$

Here we choose χ as above, with the additional property that $1_K \prec \chi$. Let $\pi(h)$ be the orthogonal projection onto $\sum_1^{M(h)} A^{(j)}(h)\mathcal{H}$, $\pi'(h) = 1-\pi(h)$. Then $\pi'(h)\chi(z-P(h))^{-1}\chi$ is holomorphic for z in a neighborhood of $\Omega_6(h)$. On the other hand, we have

$$\|\pi'(h)\chi(z-P)^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le Ae^{Ah^{-n^\sharp}\log\frac{1}{hS(h)}} \tag{11.8}$$

in $\Omega_6(h)\setminus\cup_{j=1}^{M(h)}D(z_j,hS(h))$. Put

$$\Omega_5(h) = [a(h)-5h^k, b(h)+5h^k] + i[-S(h)h^{-1-n^\sharp}, S(h)]. \tag{11.9}$$

Then (11.8), the maximum principle and the upper bound on $M(h)$ imply that (11.8) holds in all of $\Omega_5(h)$, without any discs removed. (Notice that we cannot have any *connected* chain of discs $D(z_j,hS(h))$ from $\Omega_5(h)$ to $\partial\Omega_6(h)\setminus\mathbf{R}$.)

Now for $\operatorname{Im} z > 0$, we have

$$\|\pi'(h)\chi(z-P(h))^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le \frac{1}{|\operatorname{Im} z|}, \tag{11.10}$$

and together with (11.8) and Lemma 11.4, this implies:

$$\|\pi'(h)\chi(z-P(h))^{-1}\chi\|_{\mathcal{H}\to\mathcal{H}} \le \frac{B}{S(h)},\ z\in[a(h)-h^k, b(h)+h^k]. \tag{11.11}$$

Now, use the quasimodes and conclude that for $\operatorname{Im} z > 0$:

$$\pi'(h)u_j(h) = \pi'(h)\chi(z-P)^{-1}\chi(z-P)u_j,$$

where we also used that $\chi = 1$ near $\operatorname{supp} u_j$. Let z tend to E_j and obtain:

$$\|\pi' u_j\| \le \frac{B}{S(h)}R(h) \le Bh^{n^\sharp+1}.$$

In other words,

$$\pi u_j = u_j + w_j,$$

with

$$\|w_j\| \le Bh^{n^\sharp+1}.$$

It follows that

$$a_{j,k} := (\pi u_j|\pi u_k) - \delta_{j,k} = \mathcal{O}(h^{n^\sharp+1}),$$

so the matrix $1+(a_{j,k})$ is invertible and hence $(\pi u_j)_{j=1}^{m(h)}$ are linearly independent. Hence $\dim\mathcal{R}(\pi)\ge m(h)$, so $M(h)\ge m(h)$. #

12 Microlocal analysis and dynamical bounds

In this chapter, we develop some amount of analytic microlocal analysis in order to study upper bounds for the number of resonances for second order operators in the semi-classical limit. In the case the coefficients are analytic everywhere, the first and most basic result (Theorem 12.13) is that if there are no trapped classical trajectories at some positive energy, say 1, then there is a complex neighborhood of 1, independent of h, which contains no resonances when h is small enough. The second main result (Theorem 12.15) is un upper bound on the number of resonances in certain possibly h-dependent neighborhoods of 1 in terms of a certain escape function, which measures the amount of trapped trajectories. When the classical dynamics is hyperbolic this leads to a more explicit estimate in terms of the dimension of the set of trapped trajectories. (See Theorem 12.22.). The first result is implicit in Helffer-Sjöstrand [38] and is really a direct consequence of the theory developed there which is a direct microlocal treatment of resonances by means of certain FBI-transforms and suitable weighted spaces. The other two results were obtained in [72]. The theory of [38] permits to treat operators with a quite general behaviour near infinity and that was of importance in some of the main applications there. In [72] we used the same theory as the general frame-work.

Lahmar-Benbernou and Martinez [10] noticed that if we retrict the attention to operators that converge to the Laplacian near infinity, then one can use a Bargman transform (which is a special case of FBI-transforms) after a simple complex scaling, and get a simpler version of the theory. This simplifies the treatment near infinity. We follow that approach here, which permits to get faster to the main ideas of [38], [72], and especially the approach of [72] is followed without any essential changes.

12.1 Trajectories and escape functions

We will consider a simplified version of scattering for classical particles (Simon, Hunziker) and the relation with the existence of certain escape functions. We follow the appendix of [29]. Consider the classical Hamiltonian

$$p(x,\xi) = \sum_{|\alpha|\le 2} a_\alpha(x)\xi^\alpha, \tag{12.1}$$

where $a_\alpha \in C^\infty(\mathbf{R}^n;\mathbf{R})$, $\partial_x^\beta a_\alpha(x) = o_\beta(1)\langle x\rangle^{-|\beta|}$, $x \to \infty$, $\beta \neq 0$,

$$p(x,\xi) - \xi^2 \to 0,\ x \to \infty, \tag{12.2}$$

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq \frac{1}{C}|\xi|^2. \tag{12.3}$$

We shall mainly work near the energy surface $p^{-1}(1)$.

Let

$$G(x,\xi) = x\cdot\xi. \tag{12.4}$$

If $p_0(x,\xi) = \xi^2$, then $H_{p_0}G = 2\xi\cdot\partial_x(x\cdot\xi) = 2\xi^2$, which is $\geq 1/C$ on $p_0^{-1}([1-\epsilon,1+\epsilon])$ if $\epsilon \leq 1/2$. If we restrict the attention to $p^{-1}([1-\epsilon,1+\epsilon]) =: \Sigma_p^\epsilon$, then

$$|(p'_\xi - p'_{0,\xi})(x,\xi)| = o(1),\ |(p'_x - p'_{0,x})(x,\xi)| = o(1)\langle x\rangle^{-1},\ |x| \to \infty,$$

while G'_x, $\langle x\rangle^{-1}G'_\xi$ remain bounded, so

$$|(H_pG - H_{p_0}G)(x,\xi)| \to 0,\ \Sigma_p^\epsilon \ni (x,\xi) \to \infty. \tag{12.5}$$

Also note that the H_p-flow is complete in the sense that a trajectory $t \mapsto \exp(tH_p)(x,\xi)$ cannot reach infinity in finite time.

Fix $\epsilon \in]0,\frac{1}{2}]$. Clearly there is a compact set $\widetilde{K} \subset \Sigma_p^\epsilon$, such that

$$H_pG \geq \frac{1}{\widetilde{C}},\ \text{on } \Sigma_p^\epsilon \setminus \widetilde{K}, \tag{12.6}$$

for some $\widetilde{C} > 0$. Choose $T > 0$ large enough so that

$$\widetilde{K} \subset \{\rho \in \Sigma_p^\epsilon;\ 1-T < G(\rho) < T-1\}. \tag{12.7}$$

Define the outgoing (+) and incoming (–) tails:

$$\Gamma_+ = \{\rho \in \Sigma_p^\epsilon;\ \exp(tH_p) \not\to \infty,\ t \to -\infty\},$$

$$\Gamma_- = \{\rho \in \Sigma_p^\epsilon;\ \exp(tH_p) \not\to \infty,\ t \to +\infty\}.$$

Notice that if $\rho \in \Sigma_p^\epsilon \setminus \Gamma_+$, then $G(\exp tH_p(\rho)) \leq 1-T$ for some $t(\rho) < 0$, and that $G(\exp tH_p(\rho)) \searrow -\infty$, $t(\rho) > t \to -\infty$. It follows that Γ_+ is closed (and the same holds for Γ_-).

Proposition 12.1 *For every $s \in \mathbf{R}$, $\Gamma_+ \cap \{\rho \in \Sigma_p^\epsilon; G(\rho) \leq s\}$ and $\Gamma_- \cap \{\rho \in \Sigma_p^\epsilon; G(\rho) \geq s\}$ are compact.*

Proof. We only consider the first of the two sets. Notice that $\Gamma_+ \cap \{\rho \in \Sigma_p^\epsilon;\ G(\rho) \leq s\}$ is empty when $s \leq 1 - T$. Clearly there is a compact set $\widehat{K} \subset \Sigma_p^\epsilon$, depending on s, such that if $\rho \in \widetilde{K}$ and $|t| \leq (s + T - 1)\widetilde{C}$, then $\exp tH_p(\rho) \in \widehat{K}$. Then if $\mu \in \Sigma_p^\epsilon \setminus \widehat{K}$, $G(\mu) \leq s$, we know that $\exp(tH_p)(\mu)$ reaches the region $G \leq 1 - T$ for some $t \in [-\widetilde{C}(s+T-1), 0]$. Hence $\mu \notin \Gamma_+$. #

Define the trapped set $K := \Gamma_+ \cap \Gamma_-$. It is easy to see that K is compact.

Proposition 12.2 *If $\Gamma_- \neq \emptyset$ (or if $\Gamma_+ \neq \emptyset$), then $K \neq \emptyset$.*

Proof. Let $\rho \in \Gamma_-$. Then $\{\exp tH_p(\rho);\ t \geq 0\}$ is contained in a compact set L, so we have $\exp t_j H_p(\rho) \to \rho_0 \in \Sigma_p^\epsilon$, for some sequence $t_j \to +\infty$. Put $\rho_j = \exp t_j H_p(\rho)$. For every $S > 0$, we have $\exp tH_p(\rho_j) \to \exp tH_p(\rho_0)$ uniformly for $|t| < S$. But $\exp tH_p(\rho_j) = \exp(t_j + t)H_p(\rho_j)$, for $|t| < S$ and j large enough so that $t_j \geq S$. Hence $\exp tH_p(\rho_0) \in L$, $|t| < S$. Since S can be chosen arbitrarily large, we conclude that $\rho_0 \in K$. #

Though we will not need it in the following, we also consider the true tails:

$$\mathcal{T}_\pm = \Gamma_\pm \setminus K.$$

On Σ_p^ϵ we have the symplectic volume which is H_p-invariant. For any $s < T - 1$, the set $\mathcal{T}_- \cap \{\rho \in \Sigma_p^\epsilon;\ G \geq s\}$ is bounded and hence of finite volume. On the other hand, for $s < 1 - T$, we have

$$\exp(tH_p)(\mathcal{T}_- \cap \{G \geq s\}) \searrow \emptyset,\ t \to +\infty.$$

Hence

$$\mathrm{Vol}\,(\mathcal{T}_- \cap \{G \geq s\}) = \mathrm{Vol}\,(\exp(tH_p)(\mathcal{T}_- \cap \{G \geq s\})) \to 0,\ t \to +\infty.$$

We get:

Proposition 12.3

$$\mathrm{Vol}\,(\mathcal{T}_\pm) = 0.$$

In [29] we also proved:

Proposition 12.4 *The following three statements are equivalent:*
(a) $\mathcal{T}_+ \neq \emptyset$
(b) $\mathcal{T}_- \neq \emptyset$
(c) If $K_\alpha = \{\rho \in \Sigma_p^\epsilon; \operatorname{dist}(\rho, K) \leq \alpha\}$*, then* K *and* $K_\alpha \setminus K$ *are non-empty for every* $\alpha > 0$.

Example. A potential well in an island. [[[Elaborate.]]]

We next construct a modification $\widetilde{G}$ of G with $\widetilde{G} = G$ outside a bounded set and with $H_p\widetilde{G} \geq 0$ everywhere on Σ_p^ϵ. Increasing $\widetilde{K}$ if necessary, we may assume that $K \subset \widetilde{K}$. Let $H_T = \{\rho \in \Sigma_p^\epsilon; G(\rho) = T\}$. We have a diffeomorphism $\kappa_+ : \mathbf{R} \times H_T \to \Sigma_p^\epsilon \setminus \Gamma_-$, given by $\kappa_+(t, \rho) = \exp(tH_p)(\rho)$. Similarly, we have a diffeomorphism $\kappa_- : \mathbf{R} \times H_{-T} \to \Sigma_p^\epsilon \setminus \Gamma_+$ (with the obvious definition of H_{-T}).

Choose $0 < f_+ \in C_0^\infty(\Sigma_p^\epsilon \setminus \Gamma_-)$ such that $f_+ \geq H_p(G)$ with equality in $\{G \geq T\}$ and outside a compact set in $\{-T \leq G \leq T\}$. Let $G_+ \in C^\infty(\Sigma_p^\epsilon \setminus \Gamma_-)$ be the solution of

$$H_pG_+ = f_+,\ G_{+|_{H_T}} = T.$$

Then $G_+ = G$ in $\{G \geq T\}$ and outside a compact set in $\{-T \leq G \leq T\}$. We also have $G_+ \leq G$ and choosing f_+ large enough, we may assume that

$$\limsup_{\rho \to \Gamma_- \cup H_{-T}} G_+(\rho) \leq -T. \tag{12.8}$$

Let f_-, G_- have the analogous properties.

Choose $\theta_\pm \in C^\infty(\mathbf{R}; [0, 1])$ with $1 = \theta_+ + \theta_-$, $\operatorname{supp}\theta_+ \subset]1 - T, +\infty[$, $\operatorname{supp}\theta_- \subset] - \infty, T - 1[$, and put

$$\chi_+(t) = \int_{-\infty}^t \theta_+(s)ds,\ \chi_-(t) = \int_{+\infty}^t \theta_-(s)ds.$$

We may further arrange so that $\theta_-(-t) = \theta_+(t)$. Then $\chi_\pm' = \theta_\pm$ and we have

$$\operatorname{supp}\chi_+ \subset]1 - T, +\infty[,\ \operatorname{supp}\chi_- \subset] - \infty, T - 1[,$$

$$\chi_+(t) + \chi_-(t) = t.$$

Put $\widetilde{G} = \chi_+ \circ G_+ + \chi_- \circ G_-$ and notice that $\chi_+ \circ G_+$ extends to a smooth function $(= 0)$ near $\{G \leq -T\} \cup \Gamma_-$. Similarly $\chi_- \circ G_-$ is smooth and well-defined on all of Σ_p^ϵ. By construction we have $\widetilde{G} = G$ in $\{G \leq -T\} \cup \{G \geq T\} \cup (\{-T < G < T\} \setminus \text{a compact set})$.

We have

$$H_p\widetilde{G} = (\chi_+' \circ G_+)f_+ + (\chi_-' \circ G_-)f_- \geq 0, \tag{12.9}$$

$$\nabla\widetilde{G} = (\chi_+' \circ G_+)\nabla G_+ + (\chi_-' \circ G_-)\nabla G_-, \tag{12.10}$$

and get

Proposition 12.5 *Given $G = x \cdot \xi$, we can find a new "escape" function $\widetilde{G} \in C^\infty(\mathbf{R}^{2n}; \mathbf{R})$, equal to G outside a compact set, such that on Σ_p^ϵ we have $\widetilde{G} = 0$ in a neighbohood of $K = \Gamma_+ \cap \Gamma_-$ and such that locally uniformly on Σ_p^ϵ:*

$$H_p(\widetilde{G}) \geq \frac{1}{C_0}|\nabla\widetilde{G}|. \tag{12.11}$$

We may arrange so that $H_p(\widetilde{G}) > 0$ in Σ_p^ϵ outside an arbitrarily small neighborhood of K, and so that $H_p(\widetilde{G}) > 0$ everywhere on Σ_p^ϵ in the case when $K = \emptyset$.

12.2 Bargmann transforms and pseudodifferential operators

We first review some well-known facts about Bargmann transforms (which can be viewed as linearized models for FBI-tranforms) following [78]. Let $\phi(x, y)$ be a holomorphic quadratic form (i.e. a homogenous polynomial of degree 2) on $\mathbf{C}_x^n \times \mathbf{C}_y^n$ such that

$$\det \phi_{xy}'' \neq 0,\ \operatorname{Im} \phi_{yy}'' > 0. \tag{12.12}$$

In the case of a classical Bargmann transform, we have $\phi(x, y) = \frac{i}{2}(x - y)^2$. Put

$$\Phi(x) = \sup_{y \in \mathbf{R}^n} -\operatorname{Im} \phi(x, y) \tag{12.13}$$

Notice that the supremum is attained at a unique point $y(x) \in \mathbf{R}^n$. In the special case above, we get $\Phi(x) = \frac{1}{2}(\operatorname{Im} x)^2$.

For $u \in \mathcal{S}'(\mathbf{R}^n)$, $0 < h \leq 1$, the function

$$Tu(x; h) = Ch^{-3n/4} \int e^{i\phi(x,y)/h} u(y) dy,\ x \in \mathbf{C}^n \tag{12.14}$$

is entire and satisfies $|Tu(x; h)| \leq \mathcal{O}_h(1)\langle x\rangle^{N_u} e^{\Phi(x)/h}$ for some $N_u \in \mathbf{R}$. If $u \in \mathcal{S}(\mathbf{R}^n)$, then $|Tu(x; h)| \leq \mathcal{O}_{N,h}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$ for every $N \in \mathbf{R}$.

We view T as a semiclassical Fourier integral operator with associated linear canonical transformation

$$\kappa_T : \mathbf{C}^{2n} \ni (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y)) \in \mathbf{C}^{2n}. \tag{12.15}$$

For every $x \in \mathbf{C}^n$ there is a unique point $(y(x), \eta(x)) \in \mathbf{R}^{2n}$ with $\pi_x \circ \kappa_T(y(x), \eta(x)) = x$, where π_x is the natural projection $\mathbf{C}^{2n}_{x,\xi} \to \mathbf{C}^n_x$. Indeed, $y(x)$ must be the same as above and $\eta(x) = -\phi'_y(x, y(x))$. Let $\xi(x) \in \mathbf{C}^n$ be the point with $(x, \xi(x)) = \kappa_T(y(x), \eta(x))$, so that $\xi(x) = \phi'_x(x, y(x))$.

Compare with

$$\frac{2}{i}\frac{\partial \Phi}{\partial x}(x) = \frac{2}{i}\frac{\partial}{\partial x}(-\mathrm{Im}\,\phi(x, y(x))) = \frac{2}{i}\frac{\partial}{\partial x}(-\mathrm{Im}\,\phi)_{y=y(x)} =$$
$$\frac{2}{i}\frac{\partial}{\partial x}(\frac{i}{2}(\phi(x,y) - \overline{\phi(x,y)}))_{y=y(x)} = (\frac{\partial}{\partial x}\phi)(x, y(x)) = \xi(x).$$

We conclude that

$$\kappa_T(\mathbf{R}^{2n}) = \Lambda_\Phi := \{(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)); x \in \mathbf{C}^n_x\}. \tag{12.16}$$

Since κ_T is canonical for the complex symplectic form $\sigma = \sum_1^n d\eta_j \wedge dy_j$, and $\mathbf{R}^{2n}$ is I-Lagrangian, i.e. a Lagrangian manifold for the real symplectic form $\mathrm{Im}\,\sigma$, the same holds for $\sigma_{|\Lambda_\Phi}$. Further, the real 2-form $\sigma_{|\mathbf{R}^{2n}}$ is nondegenerate, so the same holds for $\sigma_{|\Lambda_\Phi}$. We say that Λ_Φ, like $\mathbf{R}^{2n}$ is an IR-manifold: Lagrangian for $\mathrm{Im}\,\sigma$ and symplectic for the restriction of $\mathrm{Re}\,\sigma$.

What does this means in terms of Φ? In general, let $F(x)$ be a smooth real-valued function on some open set in $\mathbf{C}^n$ and define Λ_F as above. Consider the restriction to Λ_F of the fundamental one form $\xi \cdot dx$ (whose exterior differential equals σ):

$$\xi \cdot dx_{|\Lambda_F} \simeq \frac{2}{i}\frac{\partial F}{\partial x} \cdot dx \simeq \frac{2}{i}\partial F, \tag{12.17}$$
$$-\mathrm{Im}\,\xi \cdot dx_{|\Lambda_F} \simeq \frac{i}{2}(\frac{2}{i}\frac{\partial F}{\partial x} \cdot dx - \frac{2}{-i}\frac{\partial F}{\partial \overline{x}} \cdot d\overline{x}) = dF,$$

so

$$-\mathrm{Im}\,\sigma_{|\Lambda_F} = d(-\mathrm{Im}\,\xi \cdot dx)_{|\Lambda_F} \simeq d^2 F = 0.$$

Hence Λ_F is I-Lagrangian for all sufficiently smooth F. Conversely, every smooth I-Lagrangian manifold of the form $\xi = \xi(x)$ is locally of the form Λ_F for a suitable smooth function F (exercise). From (12.17) we get

$$\sigma_{|\Lambda_F} \simeq d(\frac{2}{i}\frac{\partial F}{\partial x} \cdot dx) = \frac{2}{i}\sum_{j,k}\frac{\partial^2 F}{\partial \overline{x}_j \partial x_k} d\overline{x}_j \wedge dx_k \; (= \frac{2}{i}\overline{\partial}\partial F). \tag{12.18}$$

We check that Λ_F is R-symplectic iff $\det(\frac{\partial^2 F}{\partial\overline{x}\partial x}) \neq 0$. Returning to Λ_Φ we then know that $\det \frac{\partial^2 \Phi}{\partial\overline{x}\partial x} \neq 0$. On the other hand, (12.13) tells us that $\Phi(x)$ is the supremum of a family of pluriharmonic functions, so it is plurisubharmonic and then also strictly plurisubharmonic:

$$\frac{\partial^2 \Phi}{\partial\overline{x}\partial x} > 0. \tag{12.19}$$

Remark. IR-manifolds in $\mathbf{C}^{2n}_{x,\xi}$ are totally real of maximal real dimension $2n$. (More details should be given here in later versions, and in the meanwhile we refer to [73, 78].)

We shall show that T is bounded: $L^2(\mathbf{R}^2) \to H_\Phi(\mathbf{C}^n) := \mathrm{Hol}\,(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi(x)/h}L(dx))$, where $L(dx)$ denotes the Lebesgue measure on $\mathbf{C}^n$ and $\mathrm{Hol}\,(\Omega)$ denotes the space of holomorphic functions on the open set $\Omega \subset \mathbf{C}^n$. The formal adjoint of T is given by

$$T^*v(y) = \overline{C}h^{-\frac{3n}{4}} \int e^{-i\phi^*(\overline{x},y)/h} v(x) e^{-2\Phi(x)/h} L(dx), \tag{12.20}$$

where $\phi^*(x,y) := \overline{\phi(\overline{x},\overline{y})}$ is holomorphic. Then we get for $v \in \mathrm{Hol}\,(\mathbf{C}^n)$, with $|v| \leq \mathcal{O}_{N,h}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$ for all N:

$$TT^*v(x) = |C|^2 C_\phi h^{-\frac{3n}{2}+\frac{n}{2}} \int e^{\frac{2}{h}\Psi(x,\overline{z})} u(z) e^{-\frac{2}{h}\Phi(z)} L(dz),\ C_\phi > 0, \tag{12.21}$$

where

$$\Psi(x,w) = \mathrm{vc.}_y \frac{i}{2}(\phi(x,y) - \phi^*(w,y)).$$

Here "vc.$_y$" means "critical value with respect to y of" .

$\Psi(x,w)$ is holomorphic in both variables and

$$\Psi(x,\overline{x}) = \mathrm{vc.}_y \frac{i}{2}(\phi(x,y) - \phi^*(\overline{x},y)) = \Phi(x). \tag{12.22}$$

Consider the strictly plurisubharmonic quadratic form

$$F(x,y) = \frac{1}{2}(\Phi(x)+\Phi(\overline{y})) - \mathrm{Re}\,\Psi(x,y) = \frac{1}{2}\Phi(x)+\frac{1}{2}\Phi(\overline{y})-\frac{1}{2}\Psi(x,y)-\frac{1}{2}\overline{\Psi(x,y)},$$

which vanishes on the anti-diagonal $D = \{y = \overline{x}\}$. Computing ∂_x and $\partial_{\overline{x}}$ of (12.22), we get

$$\partial_x\Psi = \partial_x\Phi,\ \partial_y\Psi = \partial_{\overline{x}}\Phi \text{ on } D.$$

It follows that $\partial_x F = 0$, $\partial_{\overline{y}} F = 0$ on D and since F is real-valued, that $\nabla F = 0$ on D. The strict plurisubharmonicity of F now implies that $F(x,y) \sim |x-\overline{y}|^2$, or in other words,

$$-\Phi(x) + 2\mathrm{Re}\,\Psi(x,\overline{z}) - \Phi(z) \sim -|x-z|^2. \tag{12.23}$$

Let Πu denote the right hand side of (12.21). Using (12.23) we see that

$$\Pi = \mathcal{O}(1) : \ L^2(e^{-2\Phi/h}L(dx)) \to \mathrm{Hol} \cap L^2(e^{-2\Phi/h}L(dx)) =: H_\Phi.$$

Taking $\partial_x \partial_{\overline{x}}$ of (12.22) and using (12.19), we get

$$\det \partial_x \partial_y \Psi(x,y) \neq 0.$$

For $\lambda > 0$, $u \in \mathrm{Hol}\,(\mathbf{C}^n)$, $|u| \leq \mathcal{O}_h(1)\langle x \rangle^{N_0} e^{\Phi(x)/h}$, consider

$$I_\lambda u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_\lambda(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) dy d\theta, \tag{12.24}$$
$$\Gamma_\lambda(x) : \ \theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(\frac{x+y}{2}) + i\frac{\lambda}{2}\overline{(x-y)}.$$

It is easy to see that the integral converges and is independent of λ (by Stokes' formula). Letting first λ become very large we can further replace Γ_λ by

$$\Gamma_{t,\lambda} : \ \theta = t\frac{2}{i}\frac{\partial \Phi}{\partial x}(\frac{x+y}{2}) + i\frac{\lambda}{2}\overline{(x-y)}, \ 0 \leq t \leq 1.$$

Then let $\lambda \to +\infty$ and consider

$$I_{0,\lambda} u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{0,\lambda}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) dy d\theta$$
$$= \frac{\lambda^n}{(2\pi h)^n} \iint e^{-\frac{\lambda}{h}|x-y|^2} u(y) L(dy) \to u(x), \ \lambda \to \infty.$$

We obtain

$$I_\lambda u(x) = u(x), \ u \in H_\Phi. \tag{12.25}$$

On the other hand, we can use the Kuranishi trick to write $I_\lambda u(x)$ as

$$I_\lambda u(x) = \frac{C}{h^n} \iint_{\widetilde{\Gamma}_\lambda(x)} e^{\frac{2}{h}(\Psi(x,z)-\Psi(y,z))} u(y) dy dz. \tag{12.26}$$

This is done by a change of variables, using that $2(\Psi(x,z)-\Psi(y,z)) = i(x-y)\cdot\widetilde{\theta}(x,y,z)$ with $\widetilde{\theta}(x,y,z)=\frac{2}{i}\Psi'_x(\frac{x+y}{2},z)$. Here $\widetilde{\Gamma}_\lambda(x)$ is the image contour in the (y,z)-variables.

If we replace $\widetilde{\Gamma}_\lambda(x)$ by the contour $z=\overline{y}$ (which can be justified by means of Stokes' formula), we get

$$\frac{C}{h^n}\iint e^{\frac{2}{h}\Psi(x,\overline{y})}u(y)e^{-\frac{2}{h}\Phi(y)}dyd\overline{y}, \tag{12.27}$$

which is a non-vanishing constant times $\Pi u(x)$ in (12.21). Both Π and (12.27) are positive semi-definite on H_Φ, so C in (12.27) is $C>0$. Adjusting the contant C in the definition of T, we get $\Pi u(x)$ precisely. Using Stokes' formula, we check quite easily that $\Pi u(x)=I_\lambda u(x)$ for $u\in \mathrm{Hol}\,(\mathbf{C}^n)$ with $|u(x)|\le C(h)\langle x\rangle^{N_0}e^{\Phi(x)/h}$ for some $N_0>0$. In view of (12.25) we have shown that

$$\Pi u = u, \tag{12.28}$$

for such functions u, and in particular for $u\in H_\Phi$. It is also clear that Π is self-adjoint in $L^2_\Phi := L^2(\mathbf{C}^n;e^{-2\Phi/h}L(dx))$. Finally Π maps the latter space into H_Φ, so

$$\Pi \text{ is the orthogonal projection} : L^2_\Phi\to H_\Phi. \tag{12.29}$$

We next claim that

$$T \text{ is unitary: } L^2(\mathbf{R}^n)\to H_\Phi. \tag{12.30}$$

We have already seen that T is bounded: $L^2(\mathbf{R}^n)\to H_\Phi$ and that $TT^*=1$ on H_Φ, so it only remains to see that T is injective: If $Tu=0$, then $(\partial_x^\alpha Tu)(0)=0$ for all $\alpha\in\mathbf{N}^n$, and hence

$$\int e^{\frac{i}{h}\phi(0,y)}p(y)u(y)dy=0,$$

for every polynomial p. If we let $\mathcal{F}$ denote the Fourier transform, then this means that

$$p(D_\eta)(\mathcal{F}(u(y)e^{\frac{i}{h}\phi(0,y)}))(0)=0,$$

for all polynomials p, and hence that $\mathcal{F}(u(y)e^{\frac{i}{h}\phi(0,y)})(\eta)=0$, since this function is entire.

We next look at the action of *pseudodifferential operators*. Let $S(\Lambda_\Phi,m)$ be the space of all C^∞-functions a on Λ_Φ such that $\partial^\alpha a=\mathcal{O}_\alpha(m)$ for every multiindex α, when identifying Λ_Φ linearly with $\mathbf{R}^{2n}$. Here m is an order function with

$$m(x) \le C\langle x-y\rangle^{N_0} m(y),\ x,y\in\Lambda_\Phi,$$

for some constants $C, N_0 > 0$. When our symbols depend on h, it is assumed that the above estimates are uniform.

If $u \in \mathcal{O}_{N,h}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$ for every N, we put

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a(\frac{x+y}{2},\theta)u(y)dyd\theta, \qquad (12.31)$$

where $\Gamma(x)$ is the contour: $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2})$. Notice that on this contour,

$$-\Phi(x) + \mathrm{Re}\,(i(x-y)\cdot\theta) + \Phi(y) = 0,$$

so the integral converges. Using this parametrization, we get

$$\mathrm{Op}_h(a)u(x) = h^{-n}\int k(x,y;h)u(y)dy,$$

where $\partial_{\overline{x}}k = \partial_{\overline{y}}k$, so by integration by parts, we see that $\mathrm{Op}_h(a)u(x)$ is holomorphic if u is.

Let $\widetilde{m}$ be a second order function on Λ_Φ. Both m and $\widetilde{m}$ will be viewed as functions on $\mathbf{C}^n_x$, by the natural identification $\pi_x : \Lambda_\Phi \to \mathbf{C}^n_x$, where $\pi_x : (x,\xi)\mapsto x$ is the projection onto the x-space. Let

$$L^2_{\Phi,\widetilde{m}} = L^2(\mathbf{C}^n; \widetilde{m}^2 e^{-2\Phi/h}L(dx)),\ H_{\Phi,\widetilde{m}} = \mathrm{Hol}(\mathbf{C}^n)\cap L^2_{\Phi,\widetilde{m}}.$$

Proposition 12.6 $\mathrm{Op}_h(a)$ *extends to a bounded operator:* $H_{\Phi,\widetilde{m}} \to H_{\Phi,\frac{\widetilde{m}}{m}}$, *whose norm is uniformly bounded with respect to* h.

Proof. For $0\le t\le 1$, let $\Gamma_t(x)$ be the contour

$$\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(\frac{x+y}{2}) + it\frac{\overline{x-y}}{\langle x-y\rangle},$$

parametrized by $y\in\mathbf{C}^n$ and let $G_{[0,1]}(x)$ be the $(n+1)$-contour obtained by letting also $t\in[0,1]$ be a parametrizing variable.

Now we have an almost holomorphic extension $\widetilde{a}$ of a to $C^\infty(\mathbf{C}^{2n})$ such that

$$\overline{\partial}\widetilde{a}(x,\xi) = \mathcal{O}_N(1)m(x)|\xi - \frac{2}{i}\frac{\partial\Phi}{\partial x}|^N,\ |\xi-\frac{2}{i}\frac{\partial\Phi}{\partial x}|\le\mathcal{O}(1),$$

for every $N \geq 0$ and such that $\widetilde{a}(x,\xi) = \mathcal{O}(1)m(x)$. For simplicity we shall write $\widetilde{a}$ instead of a. Then Stokes' formula gives

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma_1(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a(\frac{x+y}{2},\theta)u(y)dyd\theta \qquad (12.32)$$
$$+\frac{1}{(2\pi h)^n}\int_{G_{[0,1]}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\partial_{\overline{y},\overline{\theta}}(a(\frac{x+y}{2},\theta))\wedge dyd\theta.$$

For the first term to the right, we notice that along $\Gamma_1(x)$ we have $dyd\theta = \mathcal{O}(1)dyd\overline{y}$ and

$$-\Phi(x) + \mathrm{Re}\,(\frac{i}{h}(x-y)\cdot\theta) + \Phi(y) = -\frac{|x-y|^2}{\langle x-y\rangle}.$$

Hence the first term to the right in (12.32) can be written

$$\frac{m(x)}{\widetilde{m}(x)} e^{\frac{\Phi(x)}{h}} h^{-n}\int e^{-\frac{1}{h}\frac{|x-y|^2}{\langle x-y\rangle}} f(x,y;h)u(y)e^{-\frac{\Phi(y)}{h}}\widetilde{m}(y)L(dy),$$

with

$$f(x,y;h) = \mathcal{O}(1)m(\frac{x+y}{2})\frac{\widetilde{m}(x)}{m(x)\widetilde{m}(y)} \leq \mathcal{O}(1)\langle x-y\rangle^{N_0},$$

so the reduced kernel

$$h^{-n}e^{-\frac{1}{h}\frac{|x-y|^2}{\langle x-y\rangle}} f(x,y;h)$$

is of modulus

$$\leq Ch^{-n}e^{-\frac{1}{h}\frac{|x-y|^2}{\langle x-y\rangle}}\langle x-y\rangle^{N_0}$$

which is the kernel of a uniformly bounded convolution operator: $L^2 \to L^2$. Thus the first term of the right hand side of (12.32) is

$$\mathcal{O}(1) : L^2_{\Phi,\widetilde{m}} \to L^2_{\Phi,\frac{\widetilde{m}}{m}}.$$

For the last term in (12.32), we notice that along $G_{[0,1]}(x)$:

$$\partial_{\overline{y},\overline{\theta}}(a(\frac{x+y}{2},\theta))\wedge dyd\theta = \mathcal{O}_N(1)m(\frac{x+y}{2})t^N\frac{|x-y|^N}{\langle x-y\rangle^N}L(dy)dt,$$

for every $N \geq 0$.

For every t we then get the reduced operator

$$v \mapsto h^{-n} \int e^{-\frac{t}{h}\frac{|x-y|^2}{\langle x-y\rangle}} k_t(x,y;h)v(y)L(dy),$$

with

$$|k_t(x,y;h)| = \mathcal{O}_N(1)m(\frac{x+y}{2})\frac{\widetilde{m}(x)}{m(x)}\frac{1}{\widetilde{m}(y)}t^N\frac{|x-y|^N}{\langle x-y\rangle^N} \leq \mathcal{O}_N(1)t^N\frac{|x-y|^N}{\langle x-y\rangle^N}|x-y|^{N_0},$$

for some fixed $N_0 > 0$. We estimate the L^2-norm of this operator by the L^1-norm of the corresponding convolution kernel:

$$\mathcal{O}_N(1)h^{-n}\int e^{-\frac{t}{h}\frac{|x|^2}{\langle x\rangle}} t^N \frac{|x|^{N+N_0}}{\langle x\rangle^N} L(dx).$$

Cut the integral into two parts: $|x| \leq 1$, $|x| \geq 1$: For $\int_{|x|\leq 1} ...$ we get the bound:

$$\mathcal{O}(1)h^{-n}\int_0^\infty e^{-\frac{t}{2h}r^2}t^N r^{N+N_0+2n-1}dr =$$
$$\mathcal{O}(1)h^{-n}\int_0^\infty e^{-\frac{tr^2}{2h}}(\frac{tr^2}{h})^{\frac{N+N_0+2n}{2}}\frac{dr}{r}\cdot h^{\frac{N+N_0+2n}{2}}t^{N-\frac{N+N_0+2n}{2}} = \mathcal{O}(1)h^{\frac{N+N_0}{2}},$$

if N is large enough.

For $\int_{|x|\geq 1}$, we get the bound:

$$\mathcal{O}(1)h^{-n}\int_1^\infty e^{-\frac{t}{2h}r}t^N r^{N_0+2n}\frac{dr}{r},$$

and the change of variables $r = \frac{h}{t}\rho$ gives

$$\mathcal{O}(1)h^{-n}\int_{\frac{t}{h}}^\infty e^{-\frac{\rho}{2}}(\frac{h}{t}\rho)^{N_0+2n}\frac{d\rho}{\rho}t^N \leq$$
$$\mathcal{O}(1)h^{-n}\int_{\frac{t}{h}}^\infty e^{-\frac{\rho}{2}}\rho^{N_0+2n}\frac{d\rho}{\rho}h^{N_0+2n}t^{N-N_0-2n}.$$

The last integral is $\mathcal{O}((1+\frac{t}{h})^{-M})$ for every $M \geq 0$. For $t \leq h^{1/2}$, we gain as many powers of h as we like from t^{N-N_0-2n}. For $t > h^{1/2}$ the integral is $\mathcal{O}((\frac{t}{h})^{-M}) \leq \mathcal{O}(h^{M/2})$ for all M and we also gain as many powers of h as we like.

We conclude that the last term in (12.32) defines an operator which is

$$\mathcal{O}(h^\infty) : L^2_{\Phi,\widetilde{m}} \to L^2_{\Phi,\frac{\widetilde{m}}{m}}.$$

This completes the proof of the proposition. #

Exercise: Show that $T\mathcal{S}(\mathbf{R}^n) = \cap_{N\geq 0} H_{\Phi,\langle x\rangle^N}$. Hint: We already know that $T\mathcal{S} \subset \cap_{N\geq 0}....$ It then suffices to show that $T^*(\cap_{N\geq 0}...) \subset \mathcal{S}$, since $TT^* = 1$ on H_Φ.

Exercise: Show that $\cap_{N\geq 0} H_{\Phi,\langle x\rangle^N}$ is dense in H_Φ. Hint: Let $\chi \in C_0^\infty(\mathbf{C}^n)$ be equal to 1 near 0 and consider $\Pi(\chi(\frac{x}{R})u)$, $R \to \infty$.

We end this section by discussing the link with h-pseudodifferential operators on $\mathbf{R}^n$. If m is an order function on $\mathbf{R}^{2n}$ and $a \in S(\mathbf{R}^{2n}, m)$, then we can define $\mathrm{Op}_h(a) : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ and we have boundedness results in Sobolev spaces analogous to the $H_{\Phi,\widetilde{m}}$ above. A classical way of developing the theory for the Weyl quantization on $\mathbf{R}^n$ (see [21] for more details) is to observe that if $\ell(x,\xi)$ is a real linear form on $\mathbf{R}^{2n}$, then $\mathrm{Op}_h(\ell) = \ell(x, hD_x)$ and that this operator is essentially self-adjoint from $C_0^\infty(\mathbf{R}^n)$. Moreover

$$e^{-i\ell(x,hD_x)} = \mathrm{Op}_h(e^{-i\ell(x,\xi)}). \tag{12.33}$$

Imitating a proof of an important invariance property for the Weyl-quantization under conjugation by certain metaplectic operators, one then observes that if k is the linear form on Λ_Φ with

$$k \circ \kappa_T = \ell, \tag{12.34}$$

then

$$k(x, hD_x)Tu = T\ell(x, hD_x)u, \ \forall u \in \mathcal{S}. \tag{12.35}$$

Now on the transform side, we have from (12.35) and the unitarity of T, that $k(x, hD_x)$ is essentially self-adjoint on H_Φ from $T\mathcal{S}(\mathbf{R}^n)$ and we also check that

$$k(x, D_x) = \mathrm{Op}_h(k), \ e^{-ik(x,hD_x)} = \mathrm{Op}_h(e^{-ik(x,\xi)}).$$

Again from (12.35), it follows that

$$e^{-ik(x,hD_x)}T = Te^{-i\ell(x,hD_x)}. \tag{12.36}$$

If $a \in \mathcal{S}(\mathbf{R}^{2n})$ and $b \in \mathcal{S}(\Lambda_\Phi)$ are related by $b \circ \kappa_T = a$, then we can represent a as a superposition of linear exponentals of the type $e^{-i\ell(x,\xi)}$ (by

Fourier's inversion formula) and we get a corresponding superposition of b in exponentials $e^{-ik(x,\xi)}$. Passing the quantizations, we get

$$\mathrm{Op}_h(b) \circ T = T \circ \mathrm{Op}_h(a). \tag{12.37}$$

By a density argument this relation extends to the case $b \in S(\Lambda_\Phi, m)$, $a \in S(\mathbf{R}^{2n}, m \circ \kappa_T)$.

For the later applications, we will need very little calculus of pseudodifferential operators. For short presentations, see [32] for the classical way (with further references given there) with a maximal use of the method of stationary phase, and [21] for the approach sketched here with essentially no explicit use of stationary phase.

12.3 Pseudodifferential operators with holomorphic symbols

Assume that $a(x,\xi)$ is holomorphic in a tubular neighborhood $\Lambda_\Phi + W$ of Λ_Φ, where W is an open bounded neighborhood of $0 \in \mathbf{C}^{2n}$, and of class $S(m)$ there, where m is an order function defined first on Λ_Φ and then extended to a full neighborhood by putting $m(x,\xi) = m(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x))$.

In this case, we have no remainder term in a formula like (12.32) and we get

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_c(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a(\frac{x+y}{2}, \theta) u(y) dy d\theta,\ u \in H_{\Phi,\widetilde{m}}, \tag{12.38}$$

if $c > 0$ is small enough, so that $\Gamma_c(x) \subset \Lambda_\Phi + W$.

Let

$$\widetilde{\Phi}(x) = \Phi(x) + f(x), \tag{12.39}$$

where $f \in C^{1,1}(\mathbf{C}^n)$ has support in some fixed compact set and sufficiently small norm: $\|f\|_{C^{1,1}} = \|\nabla f\|_{\mathrm{Lip}} + \sup_{\mathbf{C}^n} |f(x)|$, where $\|g\|_{\mathrm{Lip}} = \sup_{x\neq y} |g(x) - g(y)|/|x-y|$. Recall that the gradient of a Lipschitz function g belongs to L^∞ and that $\|g\|_{\mathrm{Lip}} = \|\nabla g\|_{L^\infty}$ (with the right choice of pointwise norm), so the same holds for the derivatives of functions of class $C^{1,1}$.

We can define the Hilbert space $H_{\widetilde{\Phi},\widetilde{m}}$, which coincides with $H_{\Phi,\widetilde{m}}$ as a space. The corresponding norms are equivalent for every fixed h but not uniformly when $h \to 0$. We have an associated Lipschitz manifold $\Lambda_{\widetilde{\Phi}}$: $\xi =$

$\frac{2}{i}\frac{\delta\widetilde{\Phi}}{\partial x}(x)$ which coincides with Λ_Φ outside a compact set and which is close to Λ_Φ in the sense of Lipschitz graphs (since f is small in $C^{1,1}$).

If $c > 0$ is small enough, we can use Stokes' formula to replace $\Gamma_c(x)$ in (12.40) by

$$\widetilde{\Gamma}_c(x) : \theta = \frac{2}{i}\frac{\partial\widetilde{\Phi}}{\partial x}(\frac{x+y}{2}) + ic\frac{\overline{x-y}}{\langle x-y\rangle}. \tag{12.40}$$

Lemma 12.7 $\mathrm{Op}_h(a)$ *is bounded:* $H_{\widetilde{\Phi},\widetilde{m}} \to H_{\widetilde{\Phi},\widetilde{m}/m}$*, uniformly with respect to* h*.*

Proof. On the level of the amplitudes, the estimates are the same as before, so we only have to check that along $\widetilde{\Gamma}_c(x)$, we have

$$|e^{\frac{1}{h}(i(x-y)\cdot\theta - \widetilde{\Phi}(x)+\widetilde{\Phi}(y))}| \le e^{-\frac{c}{2}\frac{|x-y|^2}{\langle x-y\rangle}} \tag{12.41}$$

Along $\widetilde{\Gamma}_c(x)$, we have using notation from $\mathbf{R}^{2n} \simeq \mathbf{C}^n$:

$$\mathrm{Re}\,(i(x-y)\cdot\theta) = \langle\nabla\widetilde{\Phi}(\frac{x+y}{2}), x-y\rangle - c\frac{|x-y|^2}{\langle x-y\rangle}, \tag{12.42}$$

while Taylor's formula with integral remainder gives:

$$\widetilde{\Phi}(x) - \widetilde{\Phi}(y) - \langle\nabla\widetilde{\Phi}(\frac{x+y}{2}), x-y\rangle = \tag{12.43}$$

$$\langle\int_0^1 (1-t)(\widetilde{\Phi}''(\frac{x+y}{2}+t\frac{x-y}{2}) - \widetilde{\Phi}''(\frac{x+y}{2}-t\frac{x-y}{2}))dt\,\frac{x-y}{2}, \frac{x-y}{2}\rangle$$

$$= \langle\int_0^1 (1-t)(f''(\frac{x+y}{2}+t\frac{x-y}{2}) - f''(\frac{x+y}{2}-t\frac{x-y}{2}))dt\,\frac{x-y}{2}, \frac{x-y}{2}\rangle.$$

Since the support of f belongs to some fixed compact set, we see that

$$\int(1-t)(f''(\frac{x+y}{2}+t\frac{x-y}{2}) - f''(\frac{x+y}{2}-t\frac{x-y}{2}))dt = \mathcal{O}(1)\|f''\|_{L^\infty}/\langle x-y\rangle.$$

Combining this with (12.42), (12.43), we get

$$\mathrm{Re}\,(i(x-y)\cdot\theta) - \widetilde{\Phi}(x) + \widetilde{\Phi}(y) = -(c+\mathcal{O}(1)\|f''\|_{L^\infty})\frac{|x-y|^2}{\langle x-y\rangle}, \tag{12.44}$$

which gives (12.41), when $\|f''\|_{L^\infty}$ is small enough. #

Although not necessary in the following, we end this section with a brief discussion of other quantizations. (See [78] for more details on the holomorphic side and [21] for the real case.) For $t \in [0,1]$, put (formally):

$$\mathrm{Op}_{h,t}(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_t(x)} e^{i(x-y)\cdot\theta/h} a(tx+(1-t)y,\theta)u(y)dyd\theta, \quad (12.45)$$

where $\Gamma_t(x)$: $\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(tx+(1-t)y)$. Along Γ_t we get

$$-\Phi(x) + \mathrm{Re}\,(i(x-y)\cdot\theta) + \Phi(y) = (2t-1)\Phi(x-y).$$

Hence,

Proposition 12.8 *Let Φ be strictly convex and let $a \in S(\Lambda_\Phi, m)$. Then $\mathrm{Op}_{h,t}(a)$ is a well-defined and uniformly bounded operator: $H_{\Phi,\widetilde{m}} \to H_{\Phi,\widetilde{m}/m}$ for $0 \le t \le \frac{1}{2}$. When $0 \le t < \frac{1}{2}$ it is enough to assume that $a = \mathcal{O}(m(x))$ to have the same conclusion, and when $t = 0$ we even get a uniformly bounded operator: $L^2_{\Phi,\widetilde{m}} \to H_{\Phi,\widetilde{m}/m}$.*

Next we are interested in describing the same operator with different quantizations. This is very classical in the real case, when $\mathrm{Op}_{h,t}$ is well-defined for $0 \le t \le 1$. If $a_t \in S(\mathbf{R}^{2n}, m)$ depends smoothly on t, we get for $u \in \mathcal{S}$:

$$\begin{aligned}
&\partial_t \mathrm{Op}_{h,t}(a_t)u(x) \qquad (12.46)\\
&= \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}\partial_t(a_t(tx+(1-t)y,\theta))u(y)dyd\theta\\
&= \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}(\partial_t a_t)(tx+(1-t)y,\theta)u(y)dyd\theta +\\
&\frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}(x-y)\cdot(\partial_x a_t)(tx+(1-t)y,\theta)u(y)dyd\theta,
\end{aligned}$$

where the last term equals

$$\begin{aligned}
&\frac{1}{(2\pi h)^n}\iint (\frac{h}{i}\partial_\theta)(e^{\frac{i}{h}(x-y)\cdot\theta})\cdot\partial_x a_t(tx+(1-t)y,\theta)u(y)dyd\theta = \qquad (12.47)\\
&-\frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}(\frac{h}{i}\partial_\theta\cdot\partial_x a_t)(tx+(1-t)y,\theta)u(y)dyd\theta.
\end{aligned}$$

We conclude that $\mathrm{Op}_{h,t}(a_t)$ is independent of t if a_t fulfills the *Schrödinger type* equation

$$\partial_t a_t = \frac{h}{i}(\partial_\theta\cdot\partial_x)a_t. \quad (12.48)$$

The Cauchy problem for this equation is well-posed in the sense that if $a_s = a_{t=s}$ is given in $S(\mathbf{R}^{2n}, m)$ for some $s \in [0,1]$, then we get a unique solution $a_t \in C^\infty([0,1]; S(\mathbf{R}^{2n}, m))$.

The same discussion can be carried out in the complex framework, except that (12.48) is now of *heat equation type* (on Λ_Φ) and can be solved only in the forward time direction. Thus if $a_s \in S(\Lambda_\Phi, m)$ is given, we can find $a_t \in C^\infty([s,1]; S(\Lambda_\Phi, m))$ which is entire for $t > s$ such that $\mathrm{Op}_{h,s}(a_s) = \mathrm{Op}_{h,t}(a_t)$ for $t \geq s$. Notice that $\mathrm{Op}_{h,0}$ maps weighted L^2-spaces into weighted L^2-spaces of holomorphic functions.

We point out that the strict convexity condition on Φ in the proposition dissappears if we choose to represent our operators as in (12.26). The case $t = 0$ gives Töplitz operators.

The above calculations show that if $a \in S(\Lambda_\Phi, m)$ then for $s, t \in [0,1]$ we have

$$\mathrm{Op}_{h,t}(a) - \mathrm{Op}_{h,s}(a) = \mathcal{O}(h) : \, H_{\Phi,\widetilde{m}} \to H_{\Phi,\widetilde{m}/m}. \tag{12.49}$$

If we also know that a has a holomorphic extension of class $S(m)$ to a tubular neighborhood of Λ_Φ, then we can replace Φ by $\widetilde{\Phi} = \Phi + f$ in the above result, provided that the $C^{1,1}$-norm of f is suffiently small.

12.4 Approximation by multiplication operators

The main result of this section is

Proposition 12.9 *Let a be holomorphic and of class $S(m)$ in a tubular neighborhood of Λ_Φ. Let $\widetilde{\Phi} \in C^{1,1}$ be a small perturbation of Φ as before. Then with $\xi(x) = \frac{2}{i}\frac{\partial\widetilde{\Phi}}{\partial x}(x)$:*

$$\mathrm{Op}_h(a)u(x) = a(x,\xi(x))u(x) + \sum_1^n (\partial_{\xi_j} a(x,\xi(x)))(hD_{x_j} - \xi_j(x))u(x) + Ru(x), \tag{12.50}$$

where R is of norm $\mathcal{O}(h)$: $H_{\widetilde{\Phi},\widetilde{m}} \to L^2_{\widetilde{\Phi},\widetilde{m}/m}$.

Proof. We write $\mathrm{Op}_h(a)u(x)$ in (12.38) with the contour (12.40). We also assume that $m = \widetilde{m} = 1$ for simplicty. By Taylor expansion, we get

$$\begin{aligned} a(\frac{x+y}{2}, \theta) = \; & a(x,\xi(x)) + (\partial_\xi a)(x,\xi(x)) \cdot (\theta - \xi(x)) \\ & -\frac{1}{2}(\partial_x a)(x,\xi(x)) \cdot (x-y) + \mathcal{O}(|x-y|^2) + |\theta - \xi(x)|^2). \end{aligned} \tag{12.51}$$

Here the third term to the right gives no contribution to (12.38):

$$\iint e^{\frac{i}{h}(x-y)\cdot\theta}(x-y)\cdot(\partial_x a)(x,\xi(x))u(y)dyd\theta =$$
$$(\partial_x a(x,\xi(x))\cdot\iint \frac{h}{i}\partial_\theta(e^{\frac{i}{h}(x-y)\cdot\theta}u(y))dyd\theta = 0.$$

The first two terms to the right in (12.51) lead to the corresponding terms in (12.50). Along the contour (12.40), we have $\theta-\xi(x)=\mathcal{O}(|x-y|)$, so the remainder term in (12.51) gives contribution $\mathcal{O}(|x-y|^2)$ in (12.38) and that gives an operator R with norm $\mathcal{O}(h)$. #

We use the above result to study certain scalar products.

Proposition 12.10 *Let a be as before and let $\psi(x)$ be locally Lipschitz on $\mathbf{C}^n$ with $\psi(x),|\nabla\psi(x)|=\mathcal{O}(m_\psi(x))$, where m_ψ together with m_1,m_2 are order functions with $m_\psi m\le m_1m_2$. Then*

$$(\psi\mathrm{Op}_h(a)u|v)_{L^2_{\widetilde{\Phi}}}=\int\psi(x)a(x,\xi(x))u(x)\overline{v(x)}e^{-2\widetilde{\Phi}(x)/h}L(dx) \qquad (12.52)$$
$$+\mathcal{O}(h)\|u\|_{H_{\widetilde{\Phi},m_1}}\|v\|_{H_{\widetilde{\Phi},m_2}}$$

for $u\in H_{\widetilde{\Phi},m_1}$, $v\in H_{\widetilde{\Phi},m_2}$.

That the action of a pseudodifferential operator can be approximated by a multiplication operator with an error $\mathcal{O}(h^{1/2})$ after a suitable integral transform was used in [79] and in [19]. That the error becomes only $\mathcal{O}(h)$ for scalar products was observed in [19] and used there to give a short proof of the sharp Gårding inequality. Here we are in a more general situation, and the limited regularity in the weights is quite essential for the later applications. (We follow [74].)

Proof. Assume $m_\psi=m=m_1=m_2=1$ for simplicity. Substituting (12.50) into (12.52), we see that we only have to estimate

$$\int\psi(x)(\partial_{\xi_j}a)(x,\xi(x))(hD_{x_j}-\xi_j(x))u(x)\overline{v(x)}e^{-2\widetilde{\Phi}(x)/h}L(dx). \qquad (12.53)$$

Here $(\partial_{\xi_j}a)(x,\xi(x))$ is Lipshitz, so we can make an integration by parts in (12.53) and get $\mathcal{O}(h)\|u\|_{H_{\widetilde{\Phi}}}\|v\|_{H_{\widetilde{\Phi}}}$ plus

$$\int\psi(x)(\partial_{\xi_j}a)(x,\xi(x))u(x)\overline{v(x)}(-hD_{x_j}-\xi_j(x))(e^{-2\widetilde{\Phi}(x)/h})L(dx), \qquad (12.54)$$

which vanishes, since $(-hD_{x_j} - \xi_j(x))(e^{-2\widetilde{\Phi}(x)/h}) = 0$. We also used that $hD_{x_j}\overline{v(x)} = 0$ since v is holomorphic. #

This can be iterated. Let b be second symbol like a with associated order functions m_b and write $m = m_a$ for the order function associated to a.

Proposition 12.11 *Let $a, b, \psi, m_a, m_b, m_\psi, m_1, m_2$ be as above with $m_\psi m_a m_b \le m_1 m_2$. Then,*

$$(\psi \mathrm{Op}_h(a)u|\mathrm{Op}_h(b)v)_{L^2_{\widetilde{\Phi}}} = \tag{12.55}$$
$$\int \psi(x)a(x,\xi(x))u(x)\overline{b(x,\xi(x))v(x)}e^{-2\widetilde{\Phi}(x)/h}L(dx) + \mathcal{O}(h)\|u\|_{H_{\widetilde{\Phi},m_1}}\|v\|_{H_{\widetilde{\Phi},m_2}},$$

for $u \in H_{\widetilde{\Phi},m_1}$, $v \in H_{\widetilde{\Phi},m_2}$.

Corollary 12.12 *Under the appropriate part of the same assumptions, we have*

$$\|\mathrm{Op}_h(a)u\|^2_{H_{\widetilde{\Phi}}} = \|a(x,\xi(x))u(x)\|^2_{L^2_{\widetilde{\Phi}}} + \mathcal{O}(h)\|u\|^2_{H_{\widetilde{\Phi},m_a}},$$

for $u \in H_{\widetilde{\Phi},m_a}$.

12.5 Absence of resonances in the analytic non-trapping case

Let

$$P = \sum_{|\alpha|\le 2} a_\alpha(x;h)(hD_x)^\alpha : H^2(\mathbf{R}^n) \to L^2(\mathbf{R}^2) \tag{12.56}$$

satisfy the general assumptions for "Q" in Chapter 7, that we here recall:

$a_\alpha(x;h)$ is independent of h for $|\alpha| = 2$,

$a_\alpha(\cdot;h)$ is bounded in C_b^∞, uniformly with repect to h, and $a_\alpha(\cdot;h) = a_{\alpha,0}(\cdot) + \mathcal{O}(h)$ in this space, where $a_{\alpha,0}$ is independent of h .

$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \frac{1}{C}|\xi|^2$.

$\sum_{|\alpha|\le 2} a_\alpha(x;h)\xi^\alpha \to \xi^2,\ |x| \to \infty$, uniformly with respect to h.

In addition to the assumption that the coefficients extend holomorphically to some angle, we assume them to be analytic everywhere. Thus we assume

(A) There exists a constant $C > 0$ such that the coefficients a_α extend holomorphically in x to $\{x \in \mathbf{C}^n; |\mathrm{Im}\, x| < C^{-1}\langle \mathrm{Re}\, x\rangle\}$, and the relevant parts of the preceding assumptions extend to this complex domain.

Recall that if $\theta > 0$ is small enough, then the resonances of P in $e^{-i[0,2\theta[}]0,+\infty[$ are defined to be the eigenvalues in this sector of $P_{|e^{i\theta}\mathbf{R}^n}$.

Let $p(x,\xi) = \sum_{|\alpha|\le 2} a_{\alpha,0}(x)\xi^\alpha$ and make the non-trapping assumption

(NT) There are no trapped H_p-trajectories in $p^{-1}(1)$.

By a trapped trajectory we mean a (maximal) integral curve $\mathbf{R} \ni t \mapsto \exp(tH_p)(\rho_0) \in p^{-1}(1)$ such that $|\exp(tH_p)(\rho_0)| \not\to \infty$ both when $t \to -\infty$ and when $t \to +\infty$. According to section 12.1, this means that "K" there is empty, if "ϵ" there is small enough. According to Proposition 12.5,, we can find an "escape" function

$$G(x,\xi) + f(x,\xi), \tag{12.57}$$

with $f \in C_0^\infty(\mathbf{R}^{2n};\mathbf{R})$, such that

$$H_pG > 0 \text{ on } p^{-1}(1). \tag{12.58}$$

Consider the IR-manifold:

$$\Lambda_{\epsilon G}:\ \operatorname{Im}(x,\xi) = \epsilon H_G(\operatorname{Re}(x,\xi)),\ 0 < \epsilon \ll 1. \tag{12.59}$$

(We leave as an exercise to verify that this is really an IR-manifold.)

Outside a bounded set, we have $H_G = x\cdot\delta_x - \xi\cdot\partial_\xi$ and here $\Lambda_{\epsilon G}$ becomes:

$$\operatorname{Im} x = \epsilon \operatorname{Re} x,\ \operatorname{Im}\xi = -\epsilon\operatorname{Re}\xi. \tag{12.60}$$

Let $\Gamma_\epsilon \subset \mathbf{C}^n$ be given by

$$\operatorname{Im} x = \epsilon\operatorname{Re} x, \tag{12.61}$$

so that $\Gamma_\epsilon = e^{i\theta(\epsilon)}\mathbf{R}^n$, with $\theta(\epsilon) = \operatorname{arctg}\epsilon$. Then the projection of $\Lambda_{\epsilon G} \setminus$ (compact set) is equal to Γ_ϵ and we claim that $\Lambda_{\epsilon G}$ coincides with $T^*\Gamma_\epsilon$ outside a bounded set. (We view $T^*\Gamma_\epsilon$ as a subset of $\mathbf{C}^{2n}_{x,\xi}$ as in Chapter 7.) Indeed for $(x,\xi) \in \Lambda_{\epsilon G} \setminus$ (compact set), we have $\operatorname{Im}\xi = -\epsilon\operatorname{Re}\xi$ in addition to (12.61) and hence

$$\xi\cdot dx_{|\Gamma_\epsilon} = (1-i\epsilon)\operatorname{Re}\xi\cdot(1+i\epsilon)d\operatorname{Re}x = (1+\epsilon^2)\operatorname{Re}\xi\cdot d\operatorname{Re}x$$

is real. Also notice that the parametrization $\kappa : \mathbf{R}^n \ni y \mapsto (1+i\epsilon)y = x \in \Gamma_\epsilon$ induces an identification of $T^*\mathbf{R}^n = \mathbf{R}^{2n}_{x,\xi}$ and $T^*\Gamma_\epsilon$ via the map

$$(y,\eta) \mapsto (\kappa(y), {}^t d\kappa(y)^{-1}\eta) = ((1+i\epsilon)y, (1+i\epsilon)^{-1}\eta).$$

Next parametrize $\Lambda_{\epsilon G}$ by $\mathbf{R}^{2n} \ni (x,\xi) \mapsto (x,\xi) + i\epsilon H_G(x,\xi)$ and compute $p_\epsilon = p_{|\Lambda_{\epsilon G}}$ by Taylor expansion:

$$p_\epsilon \simeq p((x,\xi) + i\epsilon H_G(x,\xi)) = p(x,\xi) - i\epsilon H_p G(x,\xi) + \mathcal{O}(\epsilon^2 \langle \xi \rangle^2). \quad (12.62)$$

This estimate is uniform since

$$H_G = \mathcal{O}(\langle x \rangle)\frac{\partial}{\partial x} + \mathcal{O}(\langle \xi \rangle)\frac{\partial}{\partial \xi},\ \partial_x^\alpha \partial_\xi^\beta G = \mathcal{O}(\langle x \rangle^{1-|\alpha|}\langle \xi \rangle^{1-|\beta|})$$
$$\partial_x^\alpha \partial_\xi^\beta p = \mathcal{O}(\langle \xi \rangle^{2-|\beta|}\langle x \rangle^{-|\alpha|}),$$

and we can work in boxes $|x - x_0| < \frac{1}{2}\langle x_0 \rangle$, $|\xi - \xi_0| < \frac{1}{2}\langle \xi_0 \rangle$, using the rescaled variables (y, η), given by

$$x = x_0 + \langle x_0 \rangle y,\ \xi = \xi_0 + \langle \xi_0 \rangle \eta,$$

for which

$$\partial_y^\alpha \partial_\eta^\beta H_G = \mathcal{O}(1)\frac{\partial}{\partial y} + \mathcal{O}(1)\frac{\partial}{\partial \eta},\ \partial_y^\alpha \partial_\eta^\beta p = \mathcal{O}(\langle \xi_0 \rangle^2).$$

Using this together with (12.58), we see that for $\epsilon > 0$ small enough, we have with the parametrization used in (12.62):

$$\operatorname{Re} p_\epsilon(x,\xi) = p(x,\xi) + \mathcal{O}(\epsilon \langle \xi \rangle^2),\ \operatorname{Im} p_\epsilon(x,\xi) \le -\frac{\epsilon}{C} + C\epsilon |p(x,\xi) - 1|. \quad (12.63)$$

In particular, we have

$$|p_\epsilon(x,\xi) - 1| \ge \epsilon \langle \xi \rangle^2 / C. \quad (12.64)$$

Let $T : L^2(\mathbf{R}^n) \to H_{\Phi_0^0}(\mathbf{C}^n)$ be a Bargman transform as in (12.14). Replacing the integration contour $\mathbf{R}^n$ there by Γ_ϵ, we get a new Bargman tranform

$$T : L^2(\Gamma_\epsilon) \to H_{\Phi_\epsilon^0}(\mathbf{C}^n), \quad (12.65)$$

by identifying Γ_ϵ with $\mathbf{R}^n$ in the natural way. Viewing $\Gamma_\epsilon \subset \mathbf{C}^n$ and $T^*\Gamma_\epsilon \subset \mathbf{C}^{2n}$ as subspaces in the natural way, we justify the writing "T" rather than "T_ϵ" and we have an associated canonical tranformation

$$\kappa_T : \mathbf{C}^{2n} \to \mathbf{C}^{2n} \text{ with } \kappa_T(T^*\Gamma_\epsilon) = \Lambda_{\Phi_\epsilon^0}, \quad (12.66)$$

where $\Phi_\epsilon^0 = \sup_{y\in\Gamma_\epsilon} -\mathrm{Im}\,\phi(x,y)$. Clearly $\Phi_\epsilon^0(x) = \Phi_0^0(x) + \mathcal{O}(\epsilon)|x|^2$ is a quadratic form which is strictly plurisubharmonic.

Now recall that $\Lambda_\epsilon = \Lambda_{\epsilon G}$ coincides with $T^*\Gamma_\epsilon$ outside a bounded set and that both manifolds are ϵ-perturbations of $\mathbf{R}^{2n}$ in the natural sense. Then $\kappa_T(\Lambda_\epsilon)$ is an IR-manifold, equal to $\Lambda_{\Phi_\epsilon^0}$ outside a bounded set and ϵ-close to that manifold everywhere, so

$$\kappa_T(\Lambda_\epsilon) = \Lambda_{\Phi_\epsilon}, \tag{12.67}$$

where $\Phi_\epsilon(x) = \Phi_\epsilon^0(x) + \epsilon g(x,\epsilon)$, where g is C^∞ in both arguments and has its x-support in a fixed compact set.

Theorem 12.13 *Under the assumptions above, and in particular (A) and (NT), there exist constants $h_0, C_0 > 0$ such that P has no resonances in the disc $D(1, C_0^{-1})$, for $0 < h < h_0$.*

Corollary 12.14 *Let $V \in C^\infty(\mathbf{R}^n;\mathbf{R})$ extend to a holomorphic function in $|\mathrm{Im}\,x| < \langle \mathrm{Re}\,\rangle / C$ which tends to 0 when $|x| \to \infty$ in that domain. Then there exist $C_0, C_1 > 0$, such that $P = -\Delta + V(x)$ has no resonances in $\{z \in \mathbf{C};\, \mathrm{Re}\,z > C_1,\, \mathrm{Im}\,z > -\mathrm{Re}\,z/C_0\}$.*

Proof of the Corollary. It will follow from the proof of the theorem, that we we have uniformity with respect to additional parameters if the assumptions hold uniformly.

Consider $P - z$ with $\mathrm{Re}\,z \gg 1$, $|\mathrm{Im}\,z| < \mathrm{Re}\,z/C_0$ and write this operator as

$$(\mathrm{Re}\,z)(-h^2\Delta + \frac{1}{\mathrm{Re}\,z}V - 1 - \frac{i\mathrm{Im}\,z}{\mathrm{Re}\,z}),\ h^2 = \frac{1}{\mathrm{Re}\,z}.$$

Theorem 12.13 implies that we have no resonances when $\mathrm{Re}\,z$ is large and $\mathrm{Im}\,z/\mathrm{Re}\,z$ is small. #

Proof of the theorem. Let $\epsilon > 0$ be sufficiently small but fixed. Let

$$Q = T \circ P_{|\Gamma_\epsilon} \circ T^{-1}. \tag{12.68}$$

Then (using the same letters for symbols and operators)

$$Q = \mathrm{Op}_h(Q) = Q^w(x, hD_x; h), \tag{12.69}$$

where on the symbol level,

$$(Q \circ \kappa_T)(x,\xi;h) = P(x,\xi;h). \tag{12.70}$$

Here P denote the Weyl-symbol of our operator P which is holomorphic in a tubular neighborhood of $T^*\Gamma_\epsilon$ and satisfies $P(x,\xi;h) = p(x,\xi) + \mathcal{O}(h\langle\xi\rangle^2)$ there. (See for instance [21].) We see that $Q(\cdot;h)$ is holomorphic in a tubular neighborhood of $\Lambda_{\Phi_\epsilon^0}$ of the form $\Lambda_{\Phi_\epsilon^0} + W$, with W independent of ϵ, and in this neighborhood we have

$$Q(x,\xi;h) = \mathcal{O}(m(x,\xi)),\ Q = q + \mathcal{O}(hm), \tag{12.71}$$

where m is the order function with $m\circ\kappa_T = \langle\xi\rangle^2$, and q is given by $q\circ\kappa_T = p$.

For $\epsilon > 0$ small enough, Λ_{Φ_ϵ} is well inside $\Lambda_{\Phi_\epsilon^0} + W$, so we can consider

$$Q^w : H_{\Phi_\epsilon,m} \to H_{\Phi_\epsilon}. \tag{12.72}$$

Let $z \in D(0, C_0^{-1})$ with $C_0^{-1} \ll \epsilon$ (ϵ being fixed). Then

$$|q(x,\xi) - z| \sim m(x,\xi) \text{ on } \Lambda_{\Phi_\epsilon}, \tag{12.73}$$

and Corollary 12.12 together with (12.71) imply that

$$\|(Q^w - z)u\|^2_{H_{\Phi_\epsilon}} \geq \frac{(1 - \mathcal{O}_\epsilon(h))}{C_\epsilon} \|u\|^2_{H_{\Phi_\epsilon},m}. \tag{12.74}$$

If z is a resonance, let $v \in H^2(\Gamma_\epsilon)$ be a corresponding eigenfunction of $P_{|\Gamma_\epsilon}$, so that $(P-z)v = 0$ on Γ_ϵ. Applying T we get $u := Tv \in H_{\Phi_\epsilon,m}$, $(Q^w - z)Tv = 0$ and (12.74) implies that if $h > 0$ is small enough, then $u = Tv = 0$ and so $v = 0$ in contradiction with the assumption that z is a resonance. #

12.6 Dynamical upper bounds in terms of escape functions

We drop the non-trapping assumtion (NT), and keep all the other assumptions of the preceding section (including (A)). Let K be the union of trapped trajectories in $\Sigma_p^{\epsilon_0} = p^{-1}([1-\epsilon_0, 1+\epsilon_0])$, where $\epsilon_0 > 0$ is small and fixed. Let G be a real-valued function on $\mathbf{R}^{2n}$, such that

$$\begin{aligned} &G \text{ and } H_pG \text{ are of class } C^{1,1},\ G = x\cdot\xi \text{ for } |(x,\xi)| \text{ large}, \\ &H_pG \text{ is } \geq \mathcal{O}(1)^{-1}|\nabla G|^2 \text{ locally uniformly in } \Sigma_p^{\epsilon_0}, \\ &H_pG > 0 \text{ outside a small neighborhood of } K \text{ in } \Sigma_p^{\epsilon_0}. \end{aligned} \tag{12.75}$$

In section 12.1 we saw that such *smooth* functions always exist, but our estimates will depend on a *specific* such function, and good choices of such functions seem to require limited regularity as we shall see in section 12.7.

For $0 \le \delta \le r \le 1$, $\delta \le C_0^{-1}$, let $M(r,\delta)$ be the number of resonances of P in $D(1+ir, r+\delta)$. Let W be a small neighborhood of K in $\Sigma_p^{\epsilon_0}$, outside which H_pG is bounded from below by some positive constant. Put

$$\begin{aligned} V(r,\delta) &= \mathrm{Vol}\,(\{\rho \in W;\ (p(\rho)-1)^2 + 2H_pG + (H_pG)^2 \le 2r\delta + \delta^2\}) \\ &= \mathrm{Vol}\,(\{\rho \in W;\ |(p(\rho) - iH_pG) - (1+ir)|^2 \le (r+\delta)^2\}). \end{aligned} \tag{12.76}$$

Notice that this quantity increases when r or δ increases. Also, if

$$\widetilde{V}(r,\delta) = \mathrm{Vol}\,(\{\rho \in W;\ (p(\rho)-1)^2 + H_pG \le r\delta\}),$$

then $V(r,\delta) \le \widetilde{V}(r, 3\delta)$, $\widetilde{V}(r,\delta) \le V(r,3\delta)$.

Theorem 12.15 *For $C_1, C_0 > 0$ sufficiently large and for $C_0 h \le \delta \le r \le 1$, $\delta \le 1/C_0$, $r\delta \ge C_0 h$, we have*

$$M(r,\delta) \le CV(r, C\delta)h^{-n}. \tag{12.77}$$

Define Γ_ϵ as in (12.61) and let $T : L^2(\Gamma_\epsilon) \to H_{\Phi_\epsilon^0}(\mathbf{C}^n)$ be a Bargman transform as in section 12.5. As there, we define Φ_ϵ, so that

$$\kappa_T(\Lambda_{\epsilon G}) = \Lambda_{\Phi_\epsilon},\ \kappa_T(T^*\Gamma_\epsilon) = \Lambda_{\Phi_\epsilon^0},\ \Phi_\epsilon = \Phi_\epsilon^0 + \epsilon g(\cdot,\epsilon), \tag{12.78}$$

where $g(\cdot,\epsilon)$ is bounded in $C^{1,1}$ when ϵ varies, and has uniformly compact support. (We recall the definition of $\Lambda_{\epsilon G}$ below.)

Having already fixed a small $\epsilon_0 > 0$, we will let $\epsilon > 0$ be small and (later) fixed. Let m be the order function with $m \circ \kappa_T = \langle \xi \rangle^2$.

By abuse of notation, we let P also denote the conjugated operator $TP_{|\Gamma_\epsilon} T^{-1}$. Then (on the transform side)

$$P = \mathcal{O}(1) :\ H_{\Phi_\epsilon^0, m} \to H_{\Phi_\epsilon^0},\ H_{\Phi_\epsilon, m} \to H_{\Phi_\epsilon},$$

uniformly in ϵ, h.

Consider the Hermitian form:

$$q(u,u) = ((P-1-ir)u|(P-1-ir)u)_{H_{\Phi_\epsilon}},\ 0 < r \le 1, \tag{12.79}$$

formally equal to $(Qu|u)_{H_{\Phi_\epsilon}}$, with $Q = (P-1-ir)^*(P-1-ir)$, where $*$ indicates that we take the adjoint in the Hilbert space H_{Φ_ϵ}.

Let $\Lambda_{\epsilon G}$ be the the image of

$$\mathbf{R}^{2n} \ni (x,\xi) \mapsto (x,\xi) + i\epsilon H_G(x,\xi), \tag{12.80}$$

and identify $\mathbf{R}^{2n}$ and $\Lambda_{\epsilon G}$ by means of this map. We also identify $\Lambda_{\epsilon G}$ and Λ_{Φ_ϵ} by means of κ_T, and Λ_{Φ_ϵ} and $\mathbf{C}^n_x$ by means of the natural projection $\pi_x : (x,\xi) \mapsto x$. Consequently we shall view $p_\epsilon := p_{|\Lambda_\epsilon}$ also as a function on $\mathbf{R}^{2n}$, Λ_{Φ_ϵ} or $\mathbf{C}^n$ whenever convenient.

According to Corollary 12.12, we have

$$q(u,u) = \int_{\mathbf{C}^n} |p_\epsilon - 1 - ir|^2 |u|^2 e^{-2\Phi_\epsilon/h} L(dx) + \mathcal{O}(h)\|u\|^2_{H_{\Phi_\epsilon},m} \tag{12.81}$$

(where p_ϵ is viewed as a function on $\mathbf{C}^n$).

If we now consider p_ϵ as a function on $\mathbf{R}^{2n}$, we know that the values of p_ϵ on the complement of $\Sigma_p^{\epsilon_0}$ and on the complement of W in $\Sigma_p^{\epsilon_0}$ avoid a set of the form $\{z;\, |\mathrm{Re}\, z - 1| < C_0^{-1},\, \mathrm{Im}\, z > -\epsilon/C_0\}$ and consequently we have here

$$|p_\epsilon - 1 - ir|^2 \geq (r + \frac{\epsilon}{C_0})^2, \tag{12.82}$$

when $\epsilon > 0$ is small enough. In the complement of $\Sigma_p^{\epsilon_0}$, we even have

$$|p_\epsilon - 1 - ir|^2 \geq r^2 + \frac{\langle \xi \rangle^2}{\mathcal{O}(1)}. \tag{12.83}$$

In a neighborhood of K in $\Sigma_p^{\epsilon_0}$, we take a closer look:

$$p_\epsilon \simeq p((x,\xi) + iH_G(x,\xi)) = p(x,\xi) - i\epsilon H_p G(x,\xi) + \mathcal{O}(\epsilon^2)|\nabla G|^2 + i\mathcal{O}(\epsilon^3)|\nabla G|^3, \tag{12.84}$$

where the first remainder term is real and the second one is purely imaginary. Hence, using (12.75):

$$-\mathrm{Im}\, p_\epsilon = \epsilon(1 + \mathcal{O}(\epsilon^2)) H_p G, \tag{12.85}$$

$$\mathrm{Re}\, p_\epsilon = p + \mathcal{O}(\epsilon^2) H_p G = p + \mathcal{O}(\epsilon)\mathrm{Im}\, p_\epsilon. \tag{12.86}$$

It follows that near K in $\Sigma_p^{\epsilon_0}$, we have

$$\begin{aligned}
|p_\epsilon - 1 - ir|^2 &= (\mathrm{Re}\, p_\epsilon - 1)^2 + (r - \mathrm{Im}\, p_\epsilon)^2 \\
&= (p - 1 + \mathcal{O}(\epsilon)\mathrm{Im}\, p_\epsilon)^2 + (r - \mathrm{Im}\, p_\epsilon)^2 \\
&\geq \frac{1}{2}(p-1)^2 - \mathcal{O}(\epsilon^2)(-\mathrm{Im}\, p_\epsilon)^2 + r^2 + 2r(-\mathrm{Im}\, p_\epsilon) + (-\mathrm{Im}\, p_\epsilon)^2 \\
&\geq r^2 + \frac{1}{2}(p-1)^2 + 2r(-\mathrm{Im}\, p_\epsilon) + \frac{1}{2}(-\mathrm{Im}\, p_\epsilon)^2 :
\end{aligned}$$

$$|p_\epsilon - 1 - ir|^2 \geq r^2 + \frac{1}{2}(p-1)^2 + 2r(-\mathrm{Im}\, p_\epsilon) + \frac{1}{2}(-\mathrm{Im}\, p_\epsilon)^2, \tag{12.87}$$

near K in $\Sigma_p^{\epsilon_0}$.

Using these estimates and a variant of (12.81), we shall first prove:

Proposition 12.16 *For* $0 \leq r \leq 1$, *we have*

$$q(u,u) \geq (r^2 - Crh)\|u\|^2_{H_{\Phi_\epsilon}}. \tag{12.88}$$

Proof. We drop the subscript H_{Φ_ϵ} for the corresponding norms and scalar products. Consider,

$$\begin{aligned} q(u,u) - r^2\|u\|^2 &= ((P-1-ir)u|(P-1-ir)u) - r^2\|u\|^2 \\ &= \|(P-1)u\|^2 + ir[((P-1)u|u) - (u|(P-1)u)] \\ &\geq r(\|(P-1)u\|^2 - 2\mathrm{Im}\,(Pu|u)) \\ &= r(\int (|p_\epsilon - 1|^2 - 2\mathrm{Im}\, p_\epsilon)|u|^2 e^{-2\Phi_\epsilon/h} L(dx) + \mathcal{O}(h)\|u\|^2_{H_{\Phi_\epsilon,m}}). \end{aligned} \tag{12.89}$$

From (12.85), (12.86) it is clear that $|p_\epsilon - 1|^2 - 2\mathrm{Im}\, p_\epsilon$ is ≥ 0 in $\Sigma_p^{\epsilon_0}$ and $\geq \mathcal{O}(1)^{-1}\langle\xi\rangle^2$ in the complement of $\Sigma_p^{\epsilon_0}$. Using this in (12.89), we get $q(u,u) - r^2\|u\|^2 \geq -Crh\|u\|^2$. #

We want to make finite rank perturbations of q for which the lower bound (12.88) improves. Fix a sufficiently small $\epsilon > 0$ for which the previous estimates hold. We start by combining (12.82), (12.83), (12.85), (12.87) to see that for $\delta > 0$ small enough, we have

$$|p_\epsilon - 1 - ir|^2 \geq r^2 + r\delta \text{ in } \mathbf{R}^{2n} \setminus W_{C\delta,r}, \tag{12.90}$$

where

$$W_{C\delta,r} = \{\rho \in \mathbf{R}^{2n};\ (p(\rho)-1)^2 + 2rH_pG + (H_pG)^2 < 2Cr\delta + (C\delta)^2\}, \tag{12.91}$$

and $C > 0$ is sufficiently large and independent of δ. (Recall that $\epsilon > 0$ has now been fixed.)

Using this in (12.81), we get

$$q(u,u) \geq (r^2 + r\delta - \mathcal{O}(h))\|u\|^2 - (r\delta + \mathcal{O}(h))\|1_{W_{C\delta,r}} u\|^2_{L^2_\Phi}, \tag{12.92}$$

where $\Phi = \Phi_\epsilon$.

We shall next show that the last term in (12.92) is very small on certain subspaces of H_Φ of finite and explicitly controled codimension. Let $\chi_0 \in C_0^\infty(B(0,1))$ (where $B(0,1)$ is the open unit ball in $\mathbf{C}^n$) be a radial function with $\int \chi_0(x)L(dx) = 1$. Let $\xi(x) = \frac{2}{i}\frac{\partial \Phi(x)}{\partial x}$. Using the mean-value property for the holomorphic function $y \mapsto e^{i(x-y)\cdot\xi(x)/h}u(y)$, we get the representation of the identity operator on H_Φ:

$$Iu(x)(=u(x)) = h^{-n}\int e^{\frac{i}{h}(x-y)\cdot\xi(x)}\chi_0(\frac{x-y}{h^{1/2}})u(y)L(dy). \qquad (12.93)$$

Let

$$K(x,y;h) = e^{\frac{i}{h}(x-y)\cdot\xi(x)}\chi_0(\frac{x-y}{h^{1/2}})h^{-n}.$$

Then the reduced kernel $e^{-(\Phi(x)-\Phi(y))/h}K(x,y;h)$ is $\mathcal{O}(1)h^{-n}\chi_0(\frac{x-y}{h^{1/2}})$ and we deduce that the last member of (12.93) defines a bounded operator: $L^2_\Phi \to L^2_\Phi$ of norm $\mathcal{O}(1)$ uniformly with respect to h.

Let x be close to $\widetilde{x}$. We shall compare the kernel $K(x,y;h)$ with the rank 1 kernel

$$\widetilde{K_{\widetilde{x}}}(x,y;h) = e^{i(x-y)\cdot\xi(\widetilde{x})/h}\chi_0(\frac{\widetilde{x}-y}{h^{1/2}})h^{-n}.$$

If $|x-\widetilde{x}| \le \epsilon h^{1/2}$ (with a new $\epsilon > 0$, the earlier one being now fixed and forgotten), we check that the difference betwen the corresponding reduced kernels is $\mathcal{O}(\epsilon h^{-n})$ and has its support in a domain $|x-y| \le \mathrm{Const.}\, h^{1/2}$.

Now let $W \subset\subset \mathbf{C}^n$ be a compact set which can be covered by $\widetilde{M}(W,\epsilon h^{1/2})$ balls, $B_j = \overline{B(x_j,\epsilon h^{1/2})}$. Let $W = \cup_{j=1}^{\widetilde{M}} W_j$ be a partition of W with $W_j \subset B_j$, $W_j \cap W_k = \emptyset$ for $j \neq k$. For $x \in W_j$, we put $\widetilde{x}(x) = x_j$. Then $W \ni x \mapsto \widetilde{x}(x)$ takes at most $\widetilde{M}(W,\epsilon h^{1/2})$ values and $|x-\widetilde{x}(x)| \le \epsilon h^{1/2}$. For $x \notin W$, we put $\chi u(x) = 0$ and for $x \in W$, we put

$$\chi u(x) = h^{-n}\int e^{\frac{i}{h}(x-y)\cdot\xi(\widetilde{x}(x))}\chi(\frac{\widetilde{x}(x)-y}{h^{1/2}})u(y)L(dy).$$

Then the discussion above gives:

Lemma 12.17 *$\chi : L^2_\Phi \to L^2_\Phi$ is of finite rank $\le \widetilde{M}(W,\epsilon h^{1/2})$ and*

$$\|1_W u - \chi u\|_{L^2_\Phi} \le C\epsilon\|u\|_\Phi,\ u \in H_\Phi(\mathbf{C}^n). \qquad (12.94)$$

We next estimate $\widetilde{M}(W,r)$:

Lemma 12.18 *Let $W \subset \mathbf{C}^n$ be compact and let $\widetilde{M}(W,r)$ be the minimal number of closed balls of radius $r > 0$, centered at points of W, that cover W. Then*

$$\mathrm{Vol}\,(W)r^{-2n} \le \mathrm{Vol}\,(B(0,1))\widetilde{M}(W,r) \le 2^{2n}\mathrm{Vol}\,(W + B(0,\frac{r}{2}))r^{-2n}. \quad (12.95)$$

Proof. Clearly

$$\mathrm{Vol}\,(W) \le \widetilde{M}(W,r)\mathrm{Vol}\,(B(0,r)) = \widetilde{M}(W,r)\mathrm{Vol}\,(B(0,1))r^{2n},$$

which gives the first inequality in (12.95). To prove the second inequality, let $A(r)$ be the maximum number of closed *disjoint* balls of radius $r/2$ that can be centered at points of W, and let $B(x_j, r/2)$, $j = 1, ..., A(r)$ be such a family of balls. Then,

$$A(r)\mathrm{Vol}\,(B(0,\frac{r}{2})) \le \mathrm{Vol}\,(W + B(0,\frac{r}{2})),$$

so that

$$A(r)\mathrm{Vol}\,(B(0,1)) \le 2^{2n}r^{-2n}\mathrm{Vol}\,(W + B(0,\frac{r}{2})). \quad (12.96)$$

The second part of (12.95) will then follow, if we prove that $\widetilde{M}(W,r) \le A(r)$, and to do so, it is enough to prove that the $\overline{B(x_j,r)}$ cover W. If that were not the case, there would be a point $x \in W$ which is not in any of the $\overline{B(x_j,r)}$ and hence $\overline{B(x,r/2)} \cap \overline{B(x_j,r/2)} = \emptyset$ for every j, which contradicts the maximality of the disjoint family $\overline{B(x_j,r/2)}$. #

Lemma 12.19 *Let $\widetilde{q} \ge 0$ be a $C^{1,1}$-function on $\mathbf{C}^n$ with $\liminf_{|x|\to\infty} \widetilde{q} > 0$. If $\lambda_0 > 0$ is sufficiently small, there exists a constant $C_1 > 0$ such that for $h \le \lambda \le \lambda_0$, we have for $0 \le \epsilon \le 1$:*

$$\mathrm{Vol}\,(W_\lambda + B(0,\epsilon h^{1/2})) \le \mathrm{Vol}\,(W_{\lambda + C_1\epsilon h^{1/2}\lambda^{1/2}}), \quad (12.97)$$

where $W_\lambda = \{x;\, \widetilde{q}(x) < \lambda\}$.

Proof. We prove the corresponding inclusions for the sets. If $x \in K_\lambda$, we have $\widetilde{q}(x) \le \lambda$ and $|\widetilde{q}'(x)| \le C\lambda^{1/2}$, so if $|y| \le \epsilon h^{1/2}$, then

$$\widetilde{q}(x+y) \le \lambda + \widetilde{C}(\lambda^{1/2}\epsilon h^{1/2} + \epsilon^2 h) \le \lambda + C_1\epsilon h^{1/2}\lambda^{1/2}.$$

#

We now return to (12.92) and apply the last three lemmas with $\widetilde{q} = (p(\rho)-1)^2 + 2rH_pG + (H_pG)^2$ in (12.91). We conclude that there exists an operator $\chi_{r,\delta} = \mathcal{O}(1) : L^2_\Phi \to L^2_\Phi$ of rank

$$\begin{aligned}&\le \mathrm{Vol}\,(W_{C\delta,r} + B(0,\epsilon h^{1/2}))\epsilon^{-2n}h^{-n}\\ &\le \mathrm{Vol}\,(W_{C\delta + C_2\delta^{1/2}r^{-1/2}\epsilon h^{1/2},r})\epsilon^{-2n}h^{-n},\end{aligned}$$

when $r\delta \ge C_0 h$, such that

$$\|(1_{W_{C\delta,r}} - \chi)u\| \le \mathcal{O}(\epsilon)\|u\|,\ u \in H_\Phi.$$

Choose $\epsilon > 0$, so that the last "$\mathcal{O}(\epsilon)$" is $\le 1/3$ and use it in (12.92):

$$q(u,u) \ge (r^2 + \frac{2r\delta}{3} - \mathcal{O}(h))\|u\|^2 - (r\delta + \mathcal{O}(h))\|\chi u\|^2,\ u \in H_{\Phi,m}. \tag{12.98}$$

Notice that

$$\mathrm{rank}\,(\chi) \le CV(r, C\delta)h^{-n}, \tag{12.99}$$

for a new sufficiently large constant $C > 0$.

Proof of Theorem 12.15. Consider still the Hermitian form (12.79):

$$q(u,u) = ((P-1-ir)u|(P-1-ir)u)_{H_\Phi},\ u \in H_{\Phi,m}.$$

From (12.81) it follows that q is a *closed* quadratic form with domain $H_{\Phi,m}$. Consequently, by the Lax-Milgram theorem, there exists a unique self-adjoint operator $Q \ge 0$ in H_Φ such that

$$\mathcal{D}(Q) \subset H_{\Phi,m}, \tag{12.100}$$

$$q(u,u) = (Qu|u),\ u \in \mathcal{D}(Q). \tag{12.101}$$

Formally, $Q = (P-1-ir)^*(P-1-ir)$. Let $\mu_1 \le \mu_2 \le ...$ be an enumeration of first all the discrete eigenvalues of $Q^{1/2}$ below the essential spectrum, repeated according to their multiplicity, and then if there are only finitely many such eigenvalues, the infimum of the essential spectrum repeated "ad eternam". Then we have the max-min formula

$$\mu_N^2 = \sup_{\substack{E \text{ is a subspace of } H_\Phi \\ \text{of codimension } \le N-1}} \inf_{\substack{u\in E \\ \|u\|=1}} q(u,u), \tag{12.102}$$

with the convention that $q(u,u) = +\infty$ if $u \in H_\Phi \setminus H_{\Phi,m}$.

Proposition 12.16 tells us that

$$\mu_1 \geq \sqrt{r^2 - Crh} \geq r - \mathcal{O}(h), \tag{12.103}$$

where we now restrict all parameters as in Theorem 12.15. On the other hand, (12.98), (12.99) show that if $E = \mathcal{N}(\chi)$ (the null-space of χ), then

$$\inf_{\substack{u\in E \\ \|u\|=1}} q(u,u) \geq r^2 + \frac{2r\delta}{3} - \mathcal{O}(h),$$

and since $\operatorname{codim}\mathcal{N}(\chi) \leq CV(r, C\delta)h^{-n}$, we conclude from (12.102), that

$$\mu_N^2 \geq r^2 + \frac{2r\delta}{3} - \mathcal{O}(h), \text{ for } N \geq CV(r, C\delta)h^{-n}. \tag{12.104}$$

Multiplying δ by a constant (affecting the "C" above), we may assume that,

$$\mu_N \geq r + 2\delta, \text{ for } N \geq CV(r, C\delta)h^{-n}. \tag{12.105}$$

Let $z_1, z_2, ...$ be the resonances of P in $D(1 + ir, r + 2/C_0)$ repeated according to their multiplicity and arranged so that $j \mapsto |z_j - 1 - ir|$. Then we have the Weyl inequalities:

$$\mu_1\mu_2...\mu_N \leq \prod_1^N |z_j - 1 - ir|, \tag{12.106}$$

viewing $z_j - 1 - ir$ as the eigenvalues of $P - 1 - ir$. Notice that these inequalities are "opposite" to the corresponding ones for a compact operator, where the singular values are enumerated in decreasing order and similarly for the moduli of the eigenvalues. See the appendix (dans la prochaine version c'est promis!) for a proof.

Let $M(r, \delta)$ be as in the theorem. If $M(r, \delta) \leq CV(r, C\delta)h^{-n}$, we are done. If not, we apply (12.106) with $N = M(r, \delta)$ and get from (12.103), (12.105), with $N = [CV(r, C\delta)h^{-n}]$:

$$\begin{aligned}
(r - Ch)^N (r + 2\delta)^{M-N} &\leq (r + \delta)^M, \\
(\frac{r + 2\delta}{r + \delta})^M &\leq (\frac{r + 2\delta}{r - Ch})^N, \\
M \log(\frac{r + 2\delta}{r + \delta}) &\leq N \log(\frac{r + 2\delta}{r - Ch}).
\end{aligned}$$

Here the logarithms are of the order of magnitude δ/r, so we get $M \leq \mathcal{O}(1)N$. #

By replacing the energy "1" be a real energy E which varies in a neighborhood of E, one obtains a variant of the main theorem. If $J \subset \mathrm{neigh}\,(1, \mathbf{R})$ is an interval and $0 < \epsilon \ll 1$, let

$$R(J, \epsilon) = \{\rho \in W;\ p(\rho) \in J,\ H_p G \leq \epsilon\}. \tag{12.107}$$

Then we have

Theorem 12.20 *We make the same assumptions as in Theorem 12.15. Then there are constants $C_0, C > 0$ such that for every subinterval I of $[1-\epsilon_0/2, 1+\epsilon_0/2]$, we have when $C_0 h \leq \delta \leq C_0^{-1}$,*

$$M_3(I, \delta) \leq C\mathrm{Vol}\,(R(I + [-C\widetilde{\delta}, C\widetilde{\delta}], C\delta))h^{-n}, \tag{12.108}$$

where $\widetilde{\delta} = \max(\delta, h^{1/2})$. Here $M_3(I, \delta)$ denotes the number of resonances in the rectangle $I - i[0, \delta[$.

In a later version of these notes we will put in the detailed proof, but in this version we only give the outline: Cover $I - i[0, \delta[$ by a "minimal" number of discs $D(E_j + ir, r + 2\delta)$ and remember the constraints $1 \geq r \geq \delta$, $r\delta \geq C_0 h$. If $\delta \geq h^{1/2}$, we take $r \sim \delta$ and if $C_0 h \leq \delta \leq h^{1/2}$, we take $r \sim h/\delta$. Then estimate the number of resonances in $D(E_j + ir, r + 2\delta)$ by $C\mathrm{Vol}\,(\{\rho \in W;\ p - iH_p G \in D(E_j + ir, r + C\delta)\}$. When summing over j we notice that every point in the lower half plane can belong to at most a fixed finite number of the discs $D(E_j + it, r + C\delta)$. #

12.7 Quick review of the case when the classical flow is hyperbolic

Recall that $K, \Gamma_\pm$ were defined in $\Sigma_p^{\epsilon_0}$ for some small fixed $\epsilon_0 > 0$. Let $\widehat{K}, \widehat{\Gamma}_\pm$ be the corresponding sets in $\Sigma_p^{2\epsilon_0}$. We make the following hyperbolicity assumptions about the classical dynamics:

(Hyp 1) In a neighborhood Ω_{ρ_0} of every point $\rho_0 \in K$, we can represent $\widehat{\Gamma}_\pm$ as a union of closed disjoint C^1-manifolds of dimension $n+1$, such that if $\rho \in \Omega_{\rho_0} \cap \widehat{\Gamma}_+$ and if $E_\rho^+ = T_\rho(\widehat{\Gamma}_{+,\rho})$, where $\widehat{\Gamma}_{+,\rho}$ is the corresponding leaf, then E_ρ^+ depends continuously on $\rho \in \Omega_{\rho_0} \cap \widehat{\Gamma}_+$, and contains $H_p(\rho)$. Same assumption about $E_\rho^- = T_\rho(\widehat{\Gamma}_{-,\rho})$.

(Hyp 2) E_ρ^+ and E_ρ^- are transversal for every $\rho \in \widehat{K}$, and are indpendent of the choice of Ω_{ρ_0}, containing ρ.

(Hyp 3) There exists a constant $C > 0$, such that with $\Phi_t = \exp tH_p$:

$$\|d\Phi_t(\nu)\| \le Ce^{-t/C}\|\nu\|,\ \nu \in T_\rho(\mathbf{R}^{2n})/E_\rho^+,\ \rho \in \widehat{K},\ t \ge 0, \tag{12.109}$$

$$\|d\Phi_{-t}(\nu)\| \le Ce^{-t/C}\|\nu\|,\ \nu \in T_\rho(\mathbf{R}^{2n})/E_\rho^-,\ \rho \in \widehat{K},\ t \ge 0, \tag{12.110}$$

where $d\Phi_t$ is considered as a map $T_\rho(\mathbf{R}^{2n})/E_\rho^\pm \to T_{\Phi_t(\rho)}(\mathbf{R}^{2n})/E_{\Phi_t(\rho)}^\pm$.

Theorem 12.21 *Under the above assumptions and after an arbitrarily small decrease of ϵ_0, we can find G as in Theorem 12.15 such that*

$$H_pG \ge \frac{1}{C}\mathrm{dist}\,(\cdot, \widehat{K})^2,\ \textit{near } K \textit{ in } \Sigma_p^{\epsilon_0} \textit{ and } > 0 \textit{ in } \Sigma_p^{\epsilon_0} \setminus K. \tag{12.111}$$

Let $L \subset \mathbf{R}^{2n}$ be bounded and put $L_\epsilon = \{\rho \in \mathbf{R}^{2n};\ \mathrm{dist}\,(\rho, L) < \epsilon\}$, for $\epsilon > 0$. We define the Minkowski codimension of L; $\mathrm{codim}\,(L)$, to be the supremum of all $d \ge 0$, such that $\mathrm{Vol}\,(L_\epsilon) \le \mathcal{O}(\epsilon^d)$ when $\epsilon \to 0$. Clearly $0 \le \mathrm{codim}\,(L) \le 2n$ when $L \ne \emptyset$. We say that L is of pure dimension if the supremum is attained, so that $\mathrm{Vol}\,(L_\epsilon) \le \mathcal{O}(\epsilon^{\mathrm{codim}\,(L)})$. Put $\dim\,(L) = 2n - \mathrm{codim}\,(L)$. The following result is a consequence of Theorem 12.20 and Theorem 12.21:

Theorem 12.22 *Under the above assumptions, let $d = \mathrm{codim}\,(\widehat{K})$ if $\widehat{K}$ is of pure codimension, and otherwise let $0 \le d < \mathrm{codim}\,\widehat{K}$. Then there is a constant $C_0 > 0$ such that for $0 < h \le C_0^{-1}$, $C_0 h \le \delta \le C_0^{-1}$, the number of resonances of P in the rectangle $]-\frac{\epsilon_0}{2}, \frac{\epsilon_0}{2}[-i[0, \delta[$ is $\le C_0\delta^{d/2}h^{-n}$.*

Remains to add to this section: The proof of Theorem 12.21 and a discussion of the example of several strictly convex obstacles and the corresponding example of approximating potentials.

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